Lesson Plan 11 - Basic Approaches & Integration by Parts 7.1, 7.2

1) Take attendance
2) Return MidTerm
3) Questions on Homework

Chapter 7.1

We're now going to discuss some basic approaches to integration.

There are some common derivative/anti-derivatives that you should know.

$\int \sec^2 x\, dx$	$\tan x + C$
$\int \sec x \tan x\, dx$	$\sec x + C$
$\int \frac{1}{\sqrt{1-x^2}}\, dx$	$\sin^{-1} x + C$
$\int \frac{1}{x\sqrt{1-x^2}}\, dx$	$\sec^{-1}\lvert x\rvert + C$
$\int \csc^2 x\, dx$	$-ctn\ x + C$
$\int \csc x \cos x\, dx$	$-\csc x + C$
$\int \frac{1}{1+x^2}\, dx$	$\tan^{-1} x + C$

Common integrals and integral tables.
Pass out a table of integrals from the back of the book.

Sometimes, you need to do some manipulation before a substitution will be obvious:
Example 1

$$\int \frac{1}{e^x + e^{-x}}\, dx$$

First multiply the top and bottom by e^x

Now we have $\int \frac{1}{e^x + e^{-x}}\, dx = \int \frac{1}{e^x + e^{-x}} \cdot \frac{e^x}{e^x}\, dx = \int \frac{e^x}{e^{2x}+1}\, dx$

Now substituting

$u = e^x$

$du = e^x dx$

$$\int \frac{e^x}{e^{2x}+1}\, dx = \int \frac{1}{u^2+1} du = \tan^{-1} u + C = \tan^{-1} e^x + C$$

It's always a good idea to split up fractions, and convert trig functions to sines and cosines:

Example 2

$$\int \frac{\cos x + \sin^3 x}{\sec x}\, dx = \int \cos^2 x\, dx + \int \sin^3 x \cos x\, dx$$

The first part we can solve using $\cos 2x = 2\cos^2 x - 1 \rightarrow \cos^2 x = \frac{1+\cos 2x}{2}$

The second integral can be solved easily with the substitution $u = \sin x$.

Example 3
Division with rational functions:

$$\int \frac{x^2+2x+1}{x+4}\, dx = \int (x-2)\, dx + \int \frac{7}{x+4}\, dx$$

Example 4
Complete the square

$$\int \frac{1}{\sqrt{-\left(x^2+8x+7\right)}}\,dx = \int \frac{1}{\sqrt{-\left(x^2+8x+16\right)+9}}\,dx = \int \frac{1}{\sqrt{9-\left(x+4\right)^2}}\,dx$$

Substituting $u = x+4$ we get

$$\int \frac{1}{\sqrt{9-\left(x+4\right)^2}}\,dx = \int \frac{1}{\sqrt{9-u^2}}\,du$$

Substituting here $u = 3v$

$$\int \frac{1}{\sqrt{9-u^2}}\,du = \int \frac{1}{3\sqrt{1-v^2}} 3\,dv = \sin^{-1}\left(\frac{x+4}{3}\right) + C$$

Example 5
Multiplying by 1

$$\int \frac{1}{1+\cos x}\,dx = \int \frac{1}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x}\,dx = \int \frac{1-\cos x}{1-\cos^2 x}\,dx = \int \frac{1-\cos x}{\sin^2 x}\,dx =$$
$$\int \frac{1}{\sin^2 x}\,dx - \int \frac{\cos x}{\sin^2 x}\,dx = \int \sec^2 x\,dx - \int \csc x \cot x\,dx = -\cot x - \csc x + C$$

7.2 Integration by parts

We start with the product rule:

$$\left[f(x)g(x)\right]' = f'(x)g(x) + f(x)g'(x)$$

We integrate both sides giving

$$\int\left[f(x)g(x)\right]' = \int f'(x)g(x) + \int f(x)g'(x)$$

On the left integration and differentiation cancel

$$f(x)g(x) = \int f'(x)g(x) + \int f(x)g'(x)$$

Subtracting the first part of the sum we get

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x)$$

So given a function broken up into a product $f(x)g'(x)$

we can transform it so that instead of integrating $f(x)g'(x)$

we integrate $f'(x)g(x)$

Whether this is a good idea or not depends on whether the 2nd function is easier to integrate than the first.

Example 1:

Find $\int xe^x dx$

We have a few choices here but thinking that $x'=1$ is pretty simple, we set

$f(x)=x \qquad g'(x)=e^x$ then

$f'(x)=1 \qquad g(x)=e^x$

So $\int xe^x dx = xe^x - \int 1\cdot e^x dx = xe^x - e^x + C$

In this case it might be good to see how this works in reverse:

$$\frac{d}{dx}xe^x - e^x + C = \left(xe^x + e^x\right) - e^x + 0 = xe^x$$

Example 2:

$\int x\sin x\,dx$

Again removing the x by setting it to $f(x)$ might work.

$f(x)=x \qquad g'(x)=\sin x$ then

$f'(x)=1 \qquad g(x)=-\cos x$

So $\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x$

Example 3:

$\int \ln x\,dx$ Here we want to get rid of $\ln x$.

$f(x)=\ln x \qquad g'(x)=1$ then

$f'(x)=\frac{1}{x} \qquad g(x)=x$

$$\int \ln x\,dx = x\ln x - \int \frac{x}{x}dx = x\ln x - x + C$$

Example 3:

$$\int \frac{x^3}{\left(1+x^2\right)^3}\,dx$$

Note that $\frac{d}{dx}\frac{1}{\left(1+x^2\right)^2}=\frac{-4x}{\left(1+x^2\right)^3}$, so let

$$f(x)=-\frac{x^2}{4} \qquad g'(x)=\frac{-4x}{\left(1+x^2\right)^3} \quad \text{then}$$

$$f'(x)=-\frac{x}{2} \qquad g(x)=\frac{1}{\left(1+x^2\right)^2}$$

$$\int \frac{x^3}{\left(1+x^2\right)^3}\,dx=-\frac{x^2}{4}\cdot\frac{1}{\left(1+x^2\right)^2}-\int\frac{-x}{2\left(1+x^2\right)^2}=\frac{-x^2}{4\left(1+x^2\right)^2}-\frac{1}{4\left(1+x^2\right)}+C$$

Example 4: Definite Integration

$$\int_0^1 \tan^{-1} x\,dx$$

$$f(x)=\tan^{-1} x \qquad g'(x)=1 \quad \text{then}$$

$$f'(x)=\frac{1}{1+x^2} \qquad g(x)=x$$

$$\int_0^1 \tan^{-1} x\,dx=\left[x\tan^{-1} x-\int\frac{x}{1+x^2}\,dx\right]_0^1=\left[x\tan^{-1} x-\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1=$$

$$\left[x\tan^{-1} x-\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1=\tan^{-1}1-\frac{1}{2}\ln 2-0+\frac{1}{2}\ln 1=\frac{\pi}{4}-\frac{\ln 2}{2}$$

Example 5: Repeated Integration by parts

$\int x^2 e^x \, dx$

$f(x) = x^2 \qquad g'(x) = e^x$ then

$f'(x) = 2x \qquad g(x) = e^x$

$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$

$\int xe^x dx$

$f(x) = x \qquad g'(x) = e^x$ then

$f'(x) = x \qquad g(x) = e^x$

$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x-1)$

$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2 e^x - 2e^x(x-1) = e^x(x^2 - 2x + 2)$

Pass out Handout 11