

Lesson Plan 12 - Trigonometric integrals and Substitution 7.3, 7.4

- 1) Take attendance
- 2) Quiz Next Tuesday
- 3) Questions on Homework

Integration of odd powers of sine and cosine

$$\int \cos^3(x) dx = \int \cos^2(x) \cos(x) dx = \int (1 - \sin^2(x)) \cos(x) dx =$$

$$\int \cos(x) dx - \int \sin^2(x) \cos(x) dx = \sin(x) - \frac{\sin^3(x)}{3} + C$$

$$\int \cos^5(x) dx = \int \cos^2(x) \cos^3(x) dx = \int (1 - \sin^2(x)) \cos^3(x) dx =$$

$$\int \cos^3(x) dx - \int \sin^2(x) \cos^3(x) dx$$

$$\int \sin^2(x) \cos^3(x) dx = \int \sin^2(x) \cos^2(x) \cos(x) dx = \int \sin^2(x) (1 - \sin^2(x)) \cos(x) dx =$$

$$\int \sin^2(x) \cos(x) - \sin^4(x) \cos(x) dx = \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C$$

$$\int \cos^5(x) dx = \sin(x) - \frac{\sin^3(x)}{3} - \left[\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} \right] + C =$$

$$\sin(x) - \frac{2\sin^3(x)}{3} + \frac{\sin^5(x)}{5} + C$$

For higher odd powers you would just repeat this.

For sine some signs change but otherwise the procedure is the same.

Even Powers of sine or cosine

Recall that $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$$\int \sin^2(x) dx = \frac{1}{2} \int 1 - \cos(2x) dx = \frac{1}{2} \left[x - \frac{\sin(2x)}{2} \right] + C = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

$$\int \sin^4(x) dx = \frac{1}{4} \int (1 - \cos(2x))^2 dx = \frac{1}{4} \int 1 - 2\cos^2(2x) + \cos^4(2x) dx$$

For integrals such as $\int \sin^n(x) \cos^m(x) dx$,

if either n or m are odd, then using the Pythagorean identity it can be converted to terms like:

$$\sin^p(x) \cos(x)$$

or

$$\cos^q(x) \sin(x)$$

which are easily integrated. If both n and m are even, then it can be converted to terms which are all even powers of a sine or cosine, which we now know how to integrate.

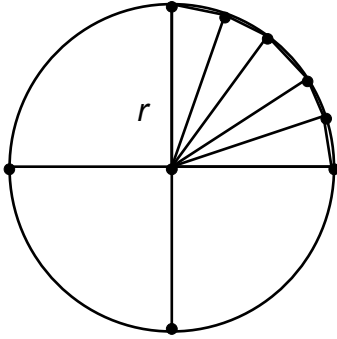
Similarly since $\sec^2(x) = 1 + \tan^2(x)$

$$\int \tan^3(x) \sec(x) dx = \int \tan(x) (\sec^2(x) + 1) \sec(x) dx =$$

$$\int \tan(x) \sec^3(x) + \int \tan(x) \sec(x) dx = \frac{\sec^3(x)}{3} + \sec(x) + C$$

Trigonometric Substitution:

Trying to integrate $\sqrt{a^2 - x^2}$ or $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ we can use a trigonometric substitution:



$$x^2 + y^2 = r^2$$

so

$$y = \pm\sqrt{r^2 - x^2}$$

$$A = 4 \int_0^r \sqrt{r^2 - x^2} dx$$

$$\text{Let } x = r \sin u \\ dx = r \cos u du$$

$$\int \sqrt{r^2 - x^2} dx = \int \sqrt{r^2 - r^2 \sin^2(u)} r \cos(u) du = \int r^2 \cos^2(u) du$$

$$4 \int_0^r \sqrt{r^2 - x^2} dx = 4r^2 \int_0^{\pi/2} \cos^2(u) du = 4r^2 \int_0^{\pi/2} \left(\frac{1 + \cos(2u)}{2} \right) du = 4r^2 \left[\frac{u + \sin(u)}{2} \right]_0^{\pi/2} =$$

$$4r^2 \left[\frac{\pi/2 + 0}{2} + 0 - 0 - \sin(\pi) \right] = \pi r^2$$

Note for $\sqrt{a^2 + x^2}$ consider the substitution $x = a \tan(u)$ since

$$a^2 + a^2 \tan^2 = a^2 (1 + \tan^2) = (a \sec)^2$$