Lesson Plan 15 - Improper Integrals 7.8

Improper Integrals

A requirement so far for evaluating definite integrals using the fundamental theorem of Calculus so far has been that the function be continuous and the limits of the integral are finite values.

We can get around this for functions with jump discontinuities by separating the integral into the sum of multiple integrals, divided at the discontinuity.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

where c is the place with the jump discontinuity occurs.

There are other integrals that we can evaluate with a little more care and work.

One example is a definite integral that is defined to infinity.

$$\int_1^\infty \frac{1}{x^2}\,dx$$

Another example is a function with a discontinuity with a limit at infinity, or one with a vertical asymptote.

$$\int_1^5 \frac{1}{\sqrt{1-x}}\,dx$$

We should note that this type of integral might have a finite value or it might diverge to infinity just as the following sequences do:

$\sum \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \ldots$ This converges to ?

on the other hand,

$\sum \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots$ this sequence diverges. That is for any large number M, there is an N such that the sum of N terms of this sum is greater than M.

So in any problem we have to consider the possibility that the integral may or may not converge to a finite value.

How do we evaluate this type of integral?

Example 1

$$\int_1^\infty \frac{1}{x^2}\,dx\,?$$

Here are the steps to evaluate this.

First we write extract the following function: $F(a) = \int_1^a \frac{1}{x^2}\,dx$

We then evaluate function at the infinite limit

$$\lim_{a\to\infty} F(x) = \lim_{a\to\infty} \int_1^a \frac{1}{x^2}\,dx = \lim_{a\to\infty}\left[-\frac{1}{x}\right]_1^a = \lim_{a\to\infty}\left[-\frac{1}{a}+1\right] = \lim_{a\to\infty} 1 - \lim_{a\to\infty}\frac{1}{a} = 1-0=1$$

We define this limit as the value of the integral. If the limit exists we call this a CONVERGENT integral.

Note that we cannot simply insert ∞ as a value.

Example 2:

Now look at this integral $\int_1^\infty \frac{1}{x}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty}\left[\ln x\right]_1^a = \lim_{a\to\infty}\left[\ln a - \ln 1\right] = \lim_{a\to\infty} \ln a$

But this expression has no limit, ln(x) grows larger than any finite value.

This is called a DIVERGENT integral.

Example 3:

$$\int_0^\infty \cos x\,dx = \lim_{a\to\infty}\int_0^a \cos x\,dx = \lim_{a\to\infty} \sin x\Big|_0^a = \lim_{a\to\infty} \sin a$$

The limit goes back and forth between 1 and -1, so this too is a DIVERGENT integral.

Example 4: $\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty}\int_{-t}^{0} xe^x dx$

$$f = x \quad g' = e^x$$
$$f' = 1 \quad g = e^x$$
$$\lim_{t\to-\infty}\int_{-t}^{0} xe^x dx = \lim_{t\to-\infty}\left[xe^x - \int e^x\right]_{-t}^{0} = \lim_{t\to-\infty}\left[xe^x - e^x\right]_{-t}^{0} =$$
$$\lim_{t\to-\infty}\left[-1-\left(te^t - e^t\right)\right] = \lim_{t\to-\infty} te^t + e^t - 1$$

The 2nd and third term are 0 and -1, but to evaluate the first term we need L'Hospital's rule.

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{\frac{d}{dt}t}{\frac{d}{dt}e^{-t}} = \lim_{t\to-\infty}\frac{1}{-e^{-t}} = -e^t = 0$$

So $\int_{-\infty}^{0} xe^x dx = -1$

Try problems 1-3 from the handout!

Continuing on about Improper integrals, splitting the integral:

Example 5: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Since both limits are infinite, we need to break this up into two parts:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

Each of these is evaluated as a separate limit:

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx =$$

$$\lim_{t\to-\infty} \left[\tan^{-1} x\right]_t^0 + \lim_{t\to\infty} \left[\tan^{-1} x\right]_0^t = \lim_{t\to-\infty} \left[0 - \tan^{-1} t\right] + \lim_{t\to\infty} \left[\tan^{-1} t - 0\right] = \frac{\pi}{2} + \frac{\pi}{2}$$

Discontinuous Integrals, Improper Integrals

Example 6: $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

Note that the function has an asymptote at $x=2$.

So we define the integral as follows:

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2}\int_t^5 \frac{1}{\sqrt{x-2}} = \lim_{t\to 2}\left[2\sqrt{x-2}\right]_t^5 = \lim_{t\to 2}\left[2\sqrt{3}-2\sqrt{t-2}\right] = 2\sqrt{3}$$

If the discontinuity is in the middle of the interval, the integral must be broken up into two parts as follows:

Example 7:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1} = \lim_{t\to 1}\int_0^t \frac{dx}{x-1} + \lim_{t\to 1}\int_t^3 \frac{dx}{x-1} =$$

$$\lim_{t\to 1}\left[\ln|x-1|\right]_0^t + \lim_{t\to 1}\left[\ln|x-1|\right]_t^3 = \lim_{t\to 1}\left[\ln|t-1| - \ln 1\right] + \lim_{t\to 1}\left[\ln|2| - \ln|t|\right]$$

But note that $\lim_{t\to 1}\ln|x-1|$ is divergent, so the integral is divergent.

Warning: If you don't divide the integral up properly you can get a false answer:

$$\int_0^3 \frac{dx}{x-1} = \left[\ln|x-1|\right]_0^3 = \ln 2 - \ln 1 = \ln 2 \quad \text{WRONG!}$$

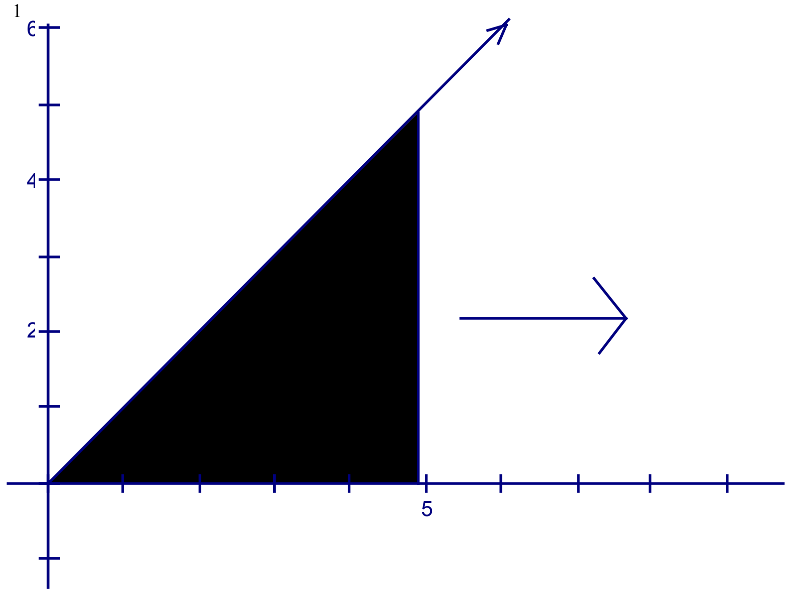
A more abstract example:

Lets look at the integral $\int_1^\infty \frac{1}{x^p}\,dx$

$$F(x)=\int_1^a \frac{1}{x^p}\,dx=\begin{cases}\frac{x^{p+1}}{p+1} & p\neq -1\\ \ln|x| & p=-1\end{cases}$$

We've already seen that $\lim_{a\to\infty}\ln|a|$ is divergent. Clearly for $p<-1$ this is also the case, eg:

$\int_1^\infty x\,dx$



So we look at $\lim_{a\to\infty}\int_1^a \frac{1}{x^p}\,dx=\lim_{a\to\infty}\frac{x^{p+1}}{p+1}\Big|_0^a=\lim_{a\to\infty}\left[\frac{a^{p+1}}{p+1}-\frac{1}{p+1}\right]$

Since $p<-1$ then $p+1<0$

Let $q=-(p+1)$ so $q>0$

So the first term is $\lim_{a\to\infty}-\frac{1}{a^q}\cdot\frac{1}{q}=0$

So for $p<-1$ $\lim_{a\to\infty}\int_1^a \frac{1}{x^p}\,dx=\frac{1}{q}$

So for $p<-1$ the integral is convergent and otherwise divergent.

Continue with handout problems.

Comparison tests with Improper integrals.

Sometimes it is important to know if an integral will converge or not, even though the function being integrated may not have an anti-derivative. An example is

$$\int_0^\infty e^{-x^2}\,dx$$

In circumstances like this, we can sometimes use the comparison test to get an answer.

The Comparison test.

Suppose that *f* and *g* are continuous functions with $f(x) \geq g(x) \geq 0$ for all $x \geq a$

Then if $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

Also if $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

It makes sense if you see this in terms of areas under a curve. A diagram here might help.

Example 9:

Show that $\int_1^\infty e^{-x^2}\,dx$ is convergent.

Note that for all $x \geq 1$ that $e^{-x} \geq e^{-x^2} \geq 0$

Also note that $\int_1^\infty e^{-x}dx = \lim_{t\to\infty}\int_1^t e^{-x}dx = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t = \lim_{t\to\infty}\left[-e^{-t}+e^{-1}\right] = \frac{1}{e}$

This proves that $\int_1^\infty e^{-x^2}\,dx$ is convergent.

Sometimes this works by comparing to a simpler function:

Example:

Show that $\int_0^\infty \frac{\sin x}{e^x}\,dx$ is convergent.

Note that for $x \geq 0$ that $\frac{\sin x}{e^x} \leq \frac{1}{e^x}$

But since $\int_0^\infty \frac{1}{e^x}\,dx$ is convergent, then $\int_0^\infty \frac{\sin x}{e^x}\,dx$ must be convergent.