Lesson Plan 18 - Hyperbolic Functions 6.10

1) Take attendance
2) Quiz
3) Bring Picture ID to FINAL!

Why learn about the hyperbolic functions?

The hyperbolic functions have an interesting set of properties and are related in a dual way to the trigonometric functions.

Both are considered "TRANSANDENTAL" functions.

This is distinguished from algebraic functions which can be described by a polynomial equation, eg:

$$y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

Transandental functions can sometimes be described by an infinite polynomial such as in the case of

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

In any case, we can easily see the relationship of the trigonometric functions with a circle by describing the circle as a curve described by parametric equations:

$$\left(\cos t, \sin t\right)$$

Likewise the hyperbolic functions

$$\sinh\left(x\right) = \frac{e^x - e^{-x}}{2}$$

$$\cosh\left(x\right) = \frac{e^x + e^{-x}}{2}$$

can describe a hyperbola as

$$\left(\cosh t, \sinh t\right)$$

Thinking about the equations of a circle and a hyperbola

$$x^2 + y^2 = 1$$

$$x^2 - y^2 = 1$$

we have the two major identities

$$\cos^2 x + \sin^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

Like the trigonometric functions there are the same combinations which give you the hyperbolic tangent, co-tangent, secant and co-secant functions, eg.

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

These functions also exhibit similar symmetry properties,

$\sinh(-x) = -\sinh(x)$ an odd function and

$\cosh(-x) = \cosh(x)$ an even function

Note that all of these properties can be derived in a straight forward manner using the known properties of $f(x) = e^x$ the exponential function which they are derived in terms of.

You might wonder at this time if the hyperbolic functions can be described so easily, why isn't there a similar definition of the trig functions?

Well there is, however it requires you to understand Complex Algebra to a higher level than we have covered so far. In particular:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Using this formula you can derive the identity $\cos^2 x + \sin^2 x = 1$ which is to say that this definition has embedded within it the Pythagorean theorem.

Other similarities found in these functions identities

Trigonometric	Hyperbolic
$1+\tan^2 x=\sec^2 x$	$1-\tanh^2 x=\sec h^2 x$
$\cos(x+y)=\cos x\cos y-\sin x\sin y$	$\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$
$\sin(x+y)=\sin x\cos y+\cos x\sin y$	$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$
$\cos(2x)=\cos^2 x-\sin^2 x$	$\cosh(2x)=\cosh^2 x+\sinh^2 x$
$\sin(2x)=2\sin x\cos x$	$\sinh(2x)=2\sinh x\cosh x$

Derivatives are easy to compute:

$$\frac{d}{dx}\sinh x=\cosh x$$

$$\frac{d}{dx}\cosh x=\sinh x$$

$$\frac{d}{dx}\tanh x=\operatorname{sech}^2 x$$

$$\frac{d}{dx}\coth x=-\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x=-\sec hx\tan x$$

$$\frac{d}{dx}\operatorname{csch} x=-\operatorname{csch} x\coth x$$

Some integrals

$$\int \sinh x\,dx=\cosh+C$$

$$\int \cosh x\,dx=\sinh+C$$

$$\int \tanh x\,dx=\ln\left|\cosh x\right|+C$$

$$\int \coth x\,dx=\ln\left|\sinh x\right|+C$$

$$\int \sec h\ x\,dx=\tan^{-1}(\sinh x)+C$$

$$\int \csc h\ x\,dx=\ln\left|\tanh(x/2)\right|+C$$

The inverse functions

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

From here it is easy to find their derivatives.

$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$$

A Catenary is the curve that a hanging rope will follow.

A catenary has the form $y = a\cosh\left(\frac{x}{a}\right)$

If a rope is held between two points 100 feet apart, the equation becomes:

$200\cosh\left(\frac{x}{200}\right)$ on the interval $[-50, 50]$

What is the length of the catenary?

From the length formula

$$L = \int_{-50}^{50} \sqrt{1+\left(f'(x)\right)^2}\,dx$$

$$f'(x) = 200\cdot\sinh\left(\frac{x}{200}\right)\cdot\frac{1}{200} = \sinh\left(\frac{x}{200}\right)$$

$$L = \int_{-50}^{50} \sqrt{1+\left(\sinh\left(\frac{x}{200}\right)\right)^2}\,dx = 2\int_{0}^{50} \sqrt{1+\left(\sinh\left(\frac{x}{200}\right)\right)^2}\,dx$$

Let $u = \frac{x}{200}$

so $du = \frac{1}{200}$

$$L = 400\int_{0}^{1/4} \sqrt{1+\left(\sinh(u)\right)^2}\,dx = 400\int_{0}^{1/4} \cosh x\,dx =$$

$$400\left[\sinh x\right]_0^{1/4} = 400\sinh\frac{1}{4} = 400\left[\frac{e^{.25}-e^{-.25}}{2}\right] \approx 101$$