Lesson Plan 5 - Integration by Substitution 5.5

1) Take attendance
2) Return Homework (Point out problem to FT
3) Questions on Homework 5.3 or 5.44 or anything else
4) Administer Quiz
5) Break

Chapter 5.5 Substitution

We start with the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ where } u = g(x)$$

in Newtonian notation

$$\left[f(g(x))\right]' = f'(g(x))g'(x)$$

Note that $du = g'(x)dx$

If we integrate both sides of the equation we get

$$\left[f(g(x))\right] = \int f'(g(x))g'(x)\,dx = \int f'(u)\,dx$$

The latter equality

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,dx$$

is called the substitution rule.
This provides us with a technique for evaluation integrals in a more direct way than the reverse engineering we've been using.

Example:

$$\int \sqrt{2x+1}\,dx$$

$$u = 2x+1 \quad \frac{du}{dx} = 2 \quad dx = \frac{du}{2}$$

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u} \cdot \frac{1}{2}du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

alternatively

$$u^2 = 2x+1 \quad u = \sqrt{2x+1} \quad \frac{du}{dx} = \frac{1}{2}\frac{2}{\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}} \quad du = \frac{dx}{\sqrt{2x+1}} \quad du = \frac{1}{u}dx \quad dx = u\,du$$

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u^2}\cdot u\,du = \int u^2 du = \frac{u^3}{3} = \frac{\left(2x+1\right)^{3/2}}{3} + C$$

Note: Not every substitution will work.

Example 2:

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx$$

Let $u = 1-4x^2$ $\quad du = -8x\,dx \quad x\,dx = -\frac{du}{8}$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{-1}{8\sqrt{u}}\,du = -\frac{1}{8}2u^{1/2} + C = -\frac{1}{4}\sqrt{1-4x^2} + C$$

Alternatively

Let $x = \frac{1}{2}\sin u$ $\quad \frac{dx}{du} = \frac{1}{2}\cos u \quad dx = \frac{1}{2}\cos u\,du$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{2}\frac{\sin u\cdot\cos u}{\sqrt{1-\sin^2 u}}\,du = \int \frac{\sin u}{2}\,du = -\frac{\cos u}{2} + C$$

But $\quad x = \frac{1}{2}\sin u \longrightarrow x^2 = \frac{\sin^2 u}{4} = \frac{1-\cos^2 u}{4} \longrightarrow \cos u = \frac{\sqrt{1-4x^2}}{2}$

So $\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\frac{\sqrt{1-4x^2}}{4} + C$

Evaluating a definite integral using substitution

Recall the first Problem

$$\int \sqrt{2x+1}\,dx = \frac{1}{3}u^{3/2} + C = \frac{1}{3}\left(2x+1\right)^{3/2} + C$$

We can evaluate the integral $\int_1^4 \sqrt{2x+1}\,dx$ in two ways

$$\left[\frac{1}{3}\left(2x+1\right)^{3/2}\right]_1^4 = \frac{1}{3}\left[9^{3/2} - 3^{3/2}\right] = 9 - \sqrt{3}$$

Or noting that when $x=1 \;\; u=3$ and when $x=4 \;\; u=9$

Plugging in $\frac{1}{3}\left[u^{3/2}\right]_3^9 = \frac{1}{3}\left[27 - 3\sqrt{3}\right] = 9 - \sqrt{3}$

This Substitution rule for definite integrals looks like this:

$$\int_a^b f'\left(g\left(x\right)\right) g'\left(x\right)\,dx = \int_{g(a)}^{g(b)} f'\left(u\right)\,dx$$

One more example:

$$\int \frac{\ln x}{x}\,dx$$

Let $u = \ln x \rightarrow du = \frac{dx}{x}$

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} = \frac{\ln^2 x}{2}$$