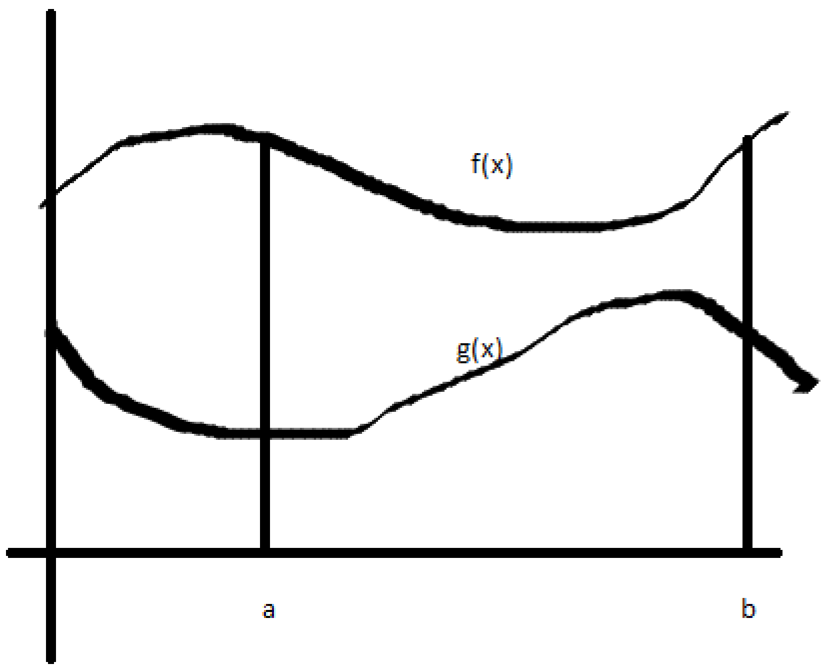
Lesson Plan 6 - Regions Between Curves 6.2

1) Take attendance
2) Return Quiz + Homework (Questions?

We've been looking at a definite integral as the area beneath a curve, that is the area between the curve and y=0.

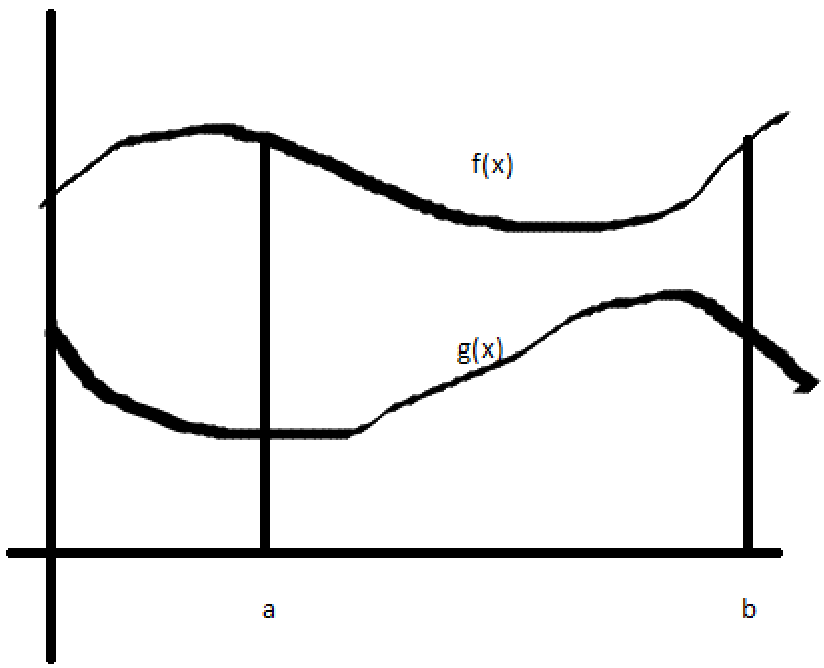
If the y coordinate of the curve is < 0 we treat this as negative area.
What about the area between two curves?



Clearly the area below f(x) minus the area below g(x) is the area between the curves.

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

What if one or both functions drop below the X axis?

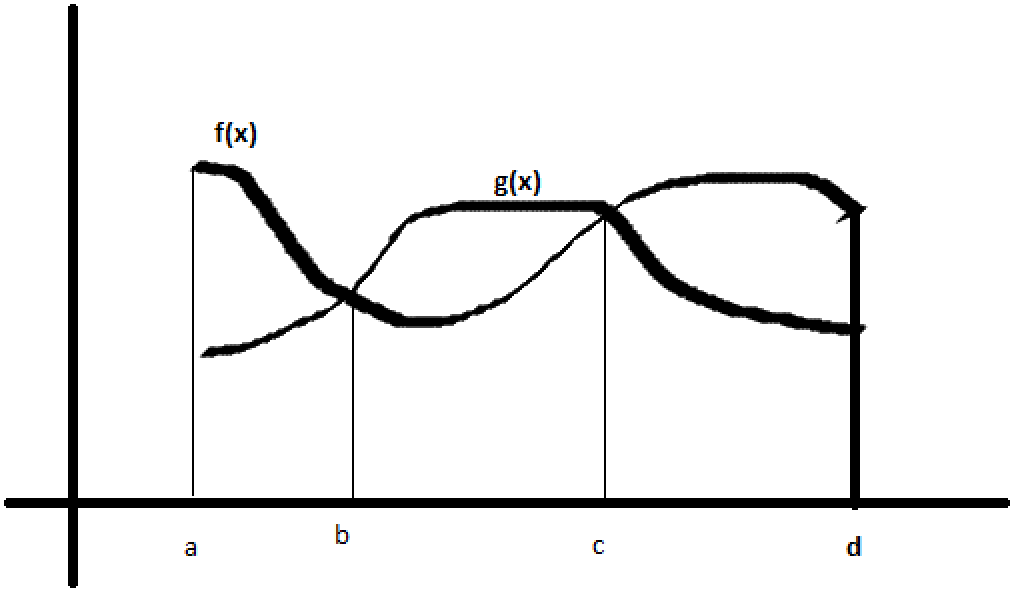


We can add a constant amount to both functions, moving them up above the X-axis preserving the area. Then:

$$\int_a^b f(x)dx + C - \left[\int_a^b g(x)dx + C \right] = \int_a^b f(x)dx - \int_a^b g(x)dx + \int_a^b C\,dx - \int_a^b C\,dx =$$

$$\int_a^b f(x)dx - \int_a^b g(x)dx$$

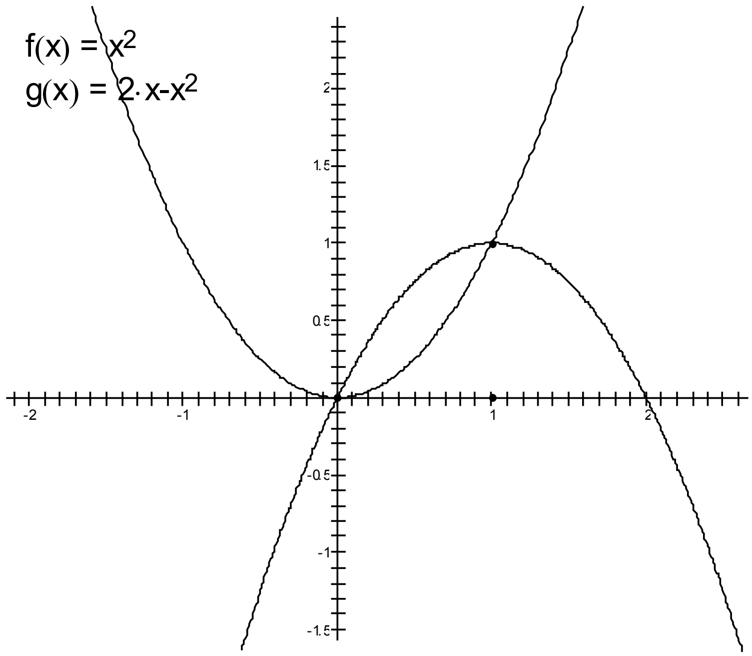
What if we have two functions that cross over and we want all the area between them?



Then we need to calculate

$$\int_a^d |f(x)-g(x)|\,dx = \int_a^b f(x)-g(x)dx - \int_b^c f(x)-g(x)dx + \int_c^d f(x)-g(x)dx \int_a^b$$

Example 1: Find the area enclosed by $y = x^2$ and $y = 2x - x^2$



Setting these equal we find

$$x^2 = 2x - x^2$$

$$2x^2 - 2x = 0$$

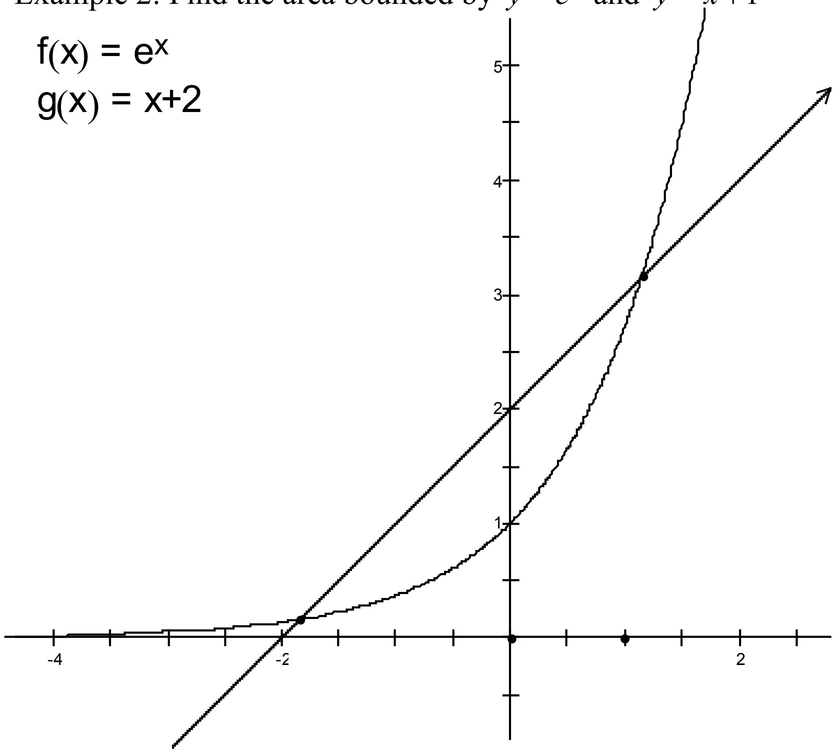
$$x(x-1) = 0$$

So the points of intersection are 0 and 1.

We integrate $\int_0^1 \left|2x - x^2 - \left(x^2\right)\right| = \int_0^1 2x - 2x^2 = \left[x^2 - \frac{2x^3}{3}\right]_0^1 = 1 - \frac{2}{3} - (0-0) = \frac{1}{3}$

Using Approximate integration

Example 2: Find the area bounded by $y = e^x$ and $y = x + 1$



We look for the points $e^x = x + 2$, but have no algebraic way to calculate the limits, so we use the calculator so solve this equation $(x+2) - e^x = 0$ using the Calc:2 Zero function.

We get values -1.841406 and 1.1461932

We could now use the anti-derivative:

$$A = \int_{-1.841406}^{1.1461932} \left(x + 2 - e^x\right) dx = \left[\frac{x^2}{2} + 2x - e^x\right]_{-1.841406}^{1.141932} =$$

$$\left[\frac{1.141932^2}{2} + 2\left(1.141932\right) - e^{1.141932}\right] - \left[\frac{-1.841406^2}{2} + 2\left(-1.841406\right) - e^{-1.841406}\right] =$$

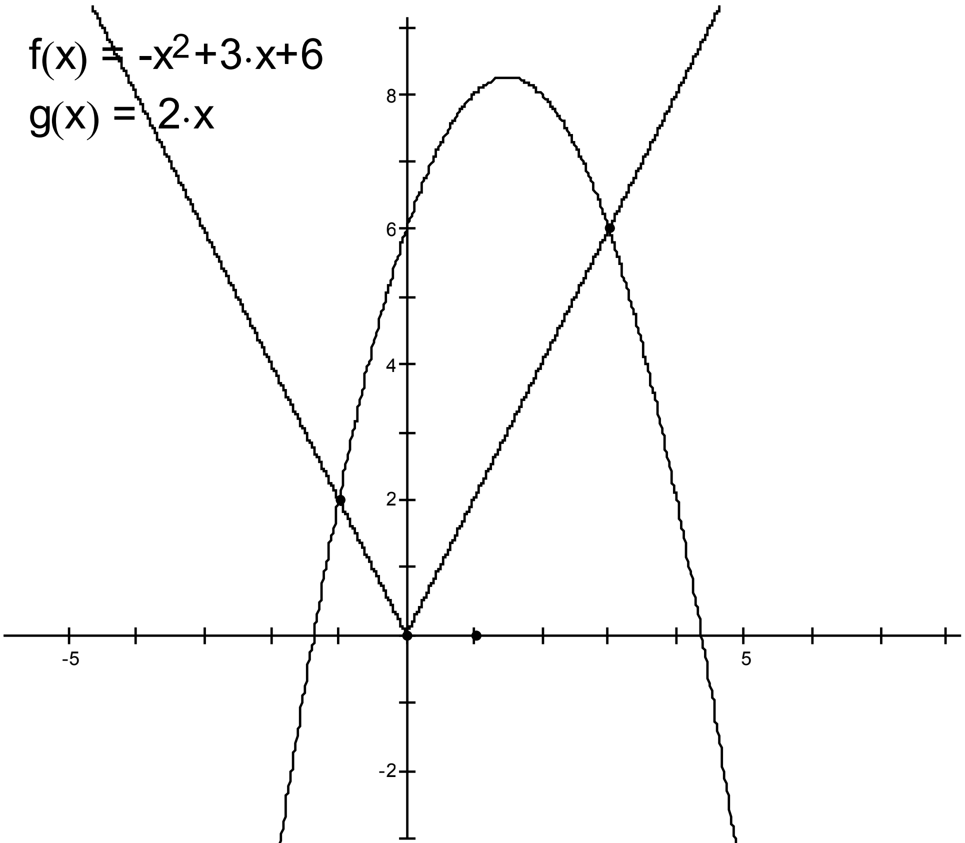
1.949071483

Or just use the Calc:7 Integration function.

1.9490715

Integrating a compound area
Example 3:

Find the area between the functions $f(x) = -x^2 + 3x + 6$ and $g(x) = |2x|$



First we want to find the interval by solving $-x^2 + 3x + 6 = -2x$ for $x < 0$

$-x^2 + 3x + 6 = -2x \rightarrow x^2 - 5x - 6 = 0 \rightarrow (x-6)(x+1) = 0$
$x = \{6, -1\}$
So the left intersection is $x = -1$

For $x > 0$
$-x^2 + 3x + 6 = 2x \rightarrow x^2 - x - 6 = 0 \rightarrow (x-3)(x+2) = 0$
$x = \{3, -2\}$
So the right intersection is $x = 3$

So we integrate $\int_{-1}^{3} -x^2+3x+6-|2x|\,dx$

Note however that

$$|2x| = \begin{cases} -2x & x<0 \\ 2x & x\geq 0 \end{cases}$$

so

$$\int_{-1}^{3} -x^2+3x+6-|2x|\,dx = \int_{-1}^{0} -x^2+3x+6+2x\,dx + \int_{0}^{3} -x^2+3x+6-2x\,dx$$

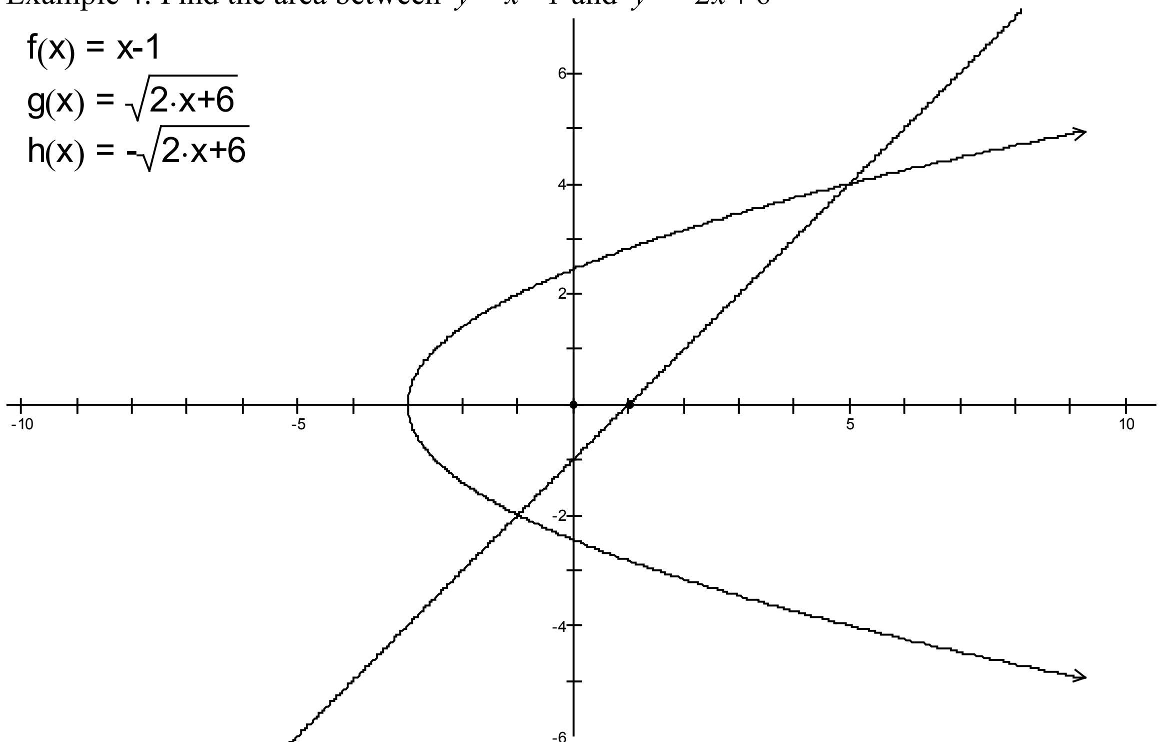
$$\int_{-1}^{0} -x^2+3x+6+2x\,dx = \left[\frac{-x^3}{3}+\frac{5x^2}{2}+6x\right]_{-1}^{0} = (0)-\left(\frac{1}{3}+\frac{5}{2}-6\right) = \left(\frac{2}{6}+\frac{15}{6}-\frac{36}{6}\right) = \frac{19}{6}$$

$$\int_{0}^{3} -x^2+3x+6-2x\,dx = \left[\frac{-x^3}{3}+\frac{x^2}{2}+6x\right]_{0}^{3} = \left(\frac{-27}{3}+\frac{9}{2}+18\right)-(0) = \left(\frac{-54}{6}+\frac{27}{6}+\frac{108}{6}\right) = \frac{81}{6}$$

$$\frac{19}{6}+\frac{81}{6} = \frac{100}{6} = \frac{50}{3}$$

Sometimes it is easier to integrate along Y instead of X?

Example 4: Find the area between $y = x - 1$ and $y^2 = 2x + 6$



Already we have a problem because the 2nd equation is not a function.

But we can switch x and y

$x = y - 1$ and $x^2 = 2y + 6$ or

$y = x + 1$ and $y = \dfrac{x^2 - 6}{2}$

We find the intersection $x + 1 = \dfrac{x^2 - 6}{2}$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$ so the interval is $[-2, 4]$

So

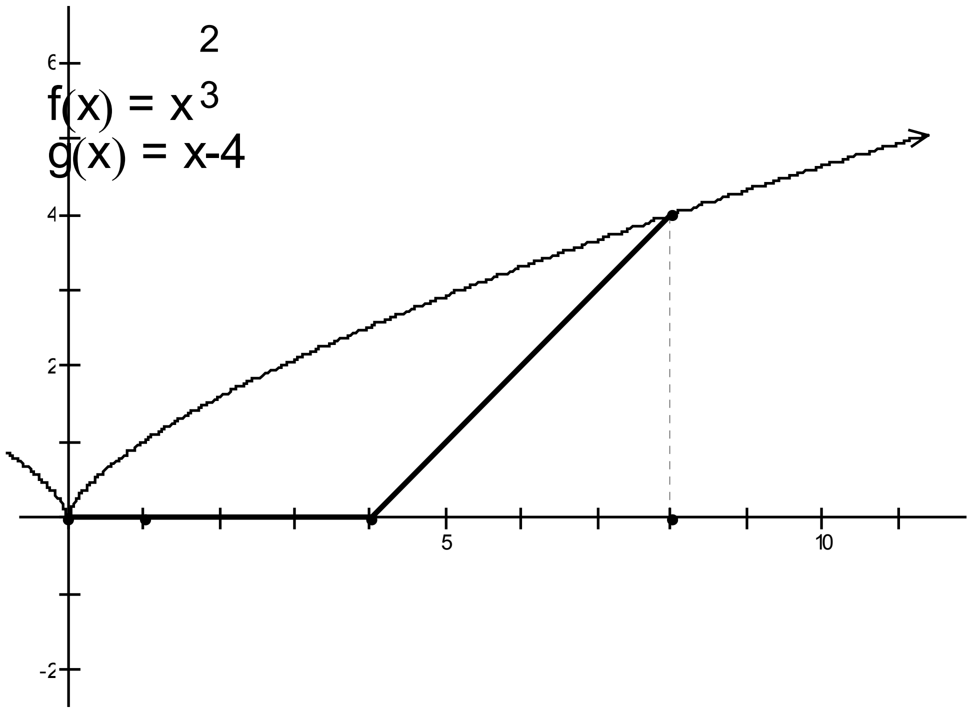
$$\int_{-2}^{4} x + 1 - \left(\frac{x^2 - 6}{2}\right) dx = \int_{-2}^{4} -\frac{x^2}{2} + x + 4\, dx = \left[-\frac{x^3}{6} + \frac{x^2}{2} + 4x\right]_{-2}^{4} =$$

$$-\frac{64}{6} + 8 + 16 - \left(\frac{-8}{6} + 2 - 8\right) = 18$$

Sometimes it helps to use some simple geometry

Example 5:

Find the area of the region in the first quadrant bounded by $y = x^{2/3}$ and $y = x - 4$



Note that we can calculate this finding

$$\int_0^8 x^{2/3}\,dx - A_{\triangle}$$

where $A_{\triangle} = \frac{1}{2} 4 \cdot 4 = 8$

$$\int_0^8 x^{2/3}\,dx - A_{\triangle} = \left[\frac{x^{5/3}}{5/3}\right]_0^8 - 8 = \frac{3}{5}[32-0] - 8 = \frac{96}{5} - \frac{40}{5} = \frac{56}{5}$$