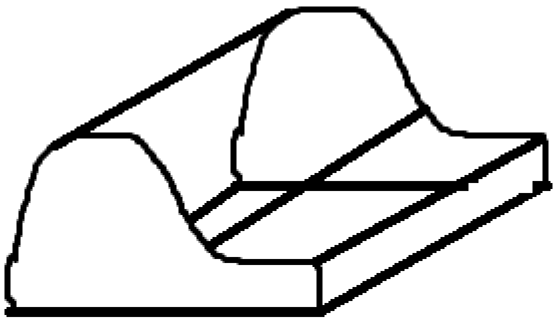
Lesson Plan 7 - 6.3 Volume by Slicing

1) Take attendance

There are different strategies for finding volumes using integrals.
If a volume has a fixed cross section:



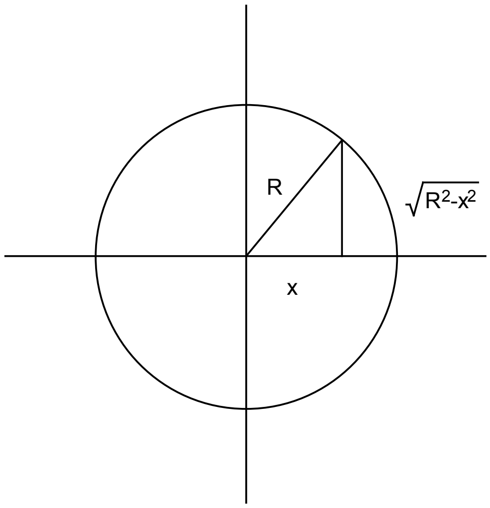
It is merely a matter of finding the area of the of the cross section and multiplying by the length.

$V = A \cdot L$

If a volume has a cross section of known area $A(x)$ at each location x then we can find the volume as follows:

$$V = \int_a^b A(x)\,dx$$

Example 1: Find the volume of a sphere.



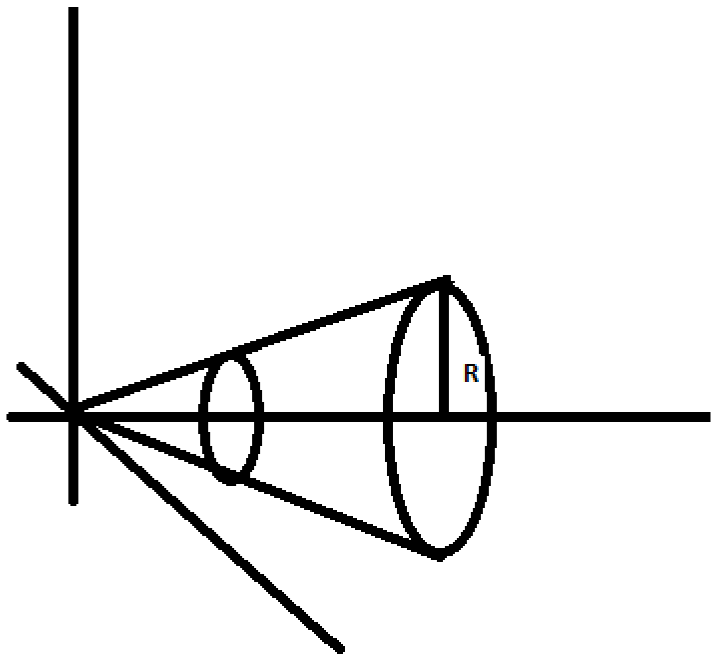
So the radius of the cross section circles is $\sqrt{R^2 - x^2}$
Since the area of a circle is πr^2 the area of each of the cross sectional circles is $\pi\left(R^2 - x^2\right)$

So our volume is

$$V = \int_{-R}^{R} \pi\left(R^2 - x^2\right) dx = \pi\left[R^2 x - \frac{x^3}{3}\right]_{-R}^{R} = \pi\left[R^3 - \frac{R^3}{3} - \left(-R^3 + \frac{R^3}{3}\right)\right] =$$

$$\pi R^3\left[2 - \frac{2}{3}\right] = \frac{4}{3}\pi R^3$$

Example 2: Cone base radius R and height L



Here the radius of each circle is $r = \frac{xR}{L}$ so the area of each circle is $A(x) = \pi \frac{x^2 R^2}{L^2}$

The integral/volume

$$V = \int_0^L A(x)\,dx = \int_0^L \pi \frac{x^2 R^2}{L^2}\,dx = \pi \frac{R^2}{L^2}\left[\frac{x^3}{3}\right]_0^L = \pi \frac{R^2}{L^2}\left[\frac{L^3}{3}\right] = \frac{1}{3}\pi R^2 L$$

The Disk Method
Example 3:

Take the function $f(x)=\sqrt{x}$ and spin it around the X axis forming a solid. What is the volume of this solid with respect to it's height.

We have $V=\int_0^H A(x)\,dx$ where

$$A(x)=\pi r^2=\pi\left(\sqrt{x}\right)^2=\pi x$$

So $V=\pi\int_0^H x\,dx=\pi\left[\frac{x^2}{2}\right]_0^H=\frac{\pi H^2}{2}$

Example 4: Rotating around the Y axis.

Let $f(x)=x^3$ be rotated around the Y axis on the interval [0,8]

$$V=\int_0^8 A(y)\,dy$$

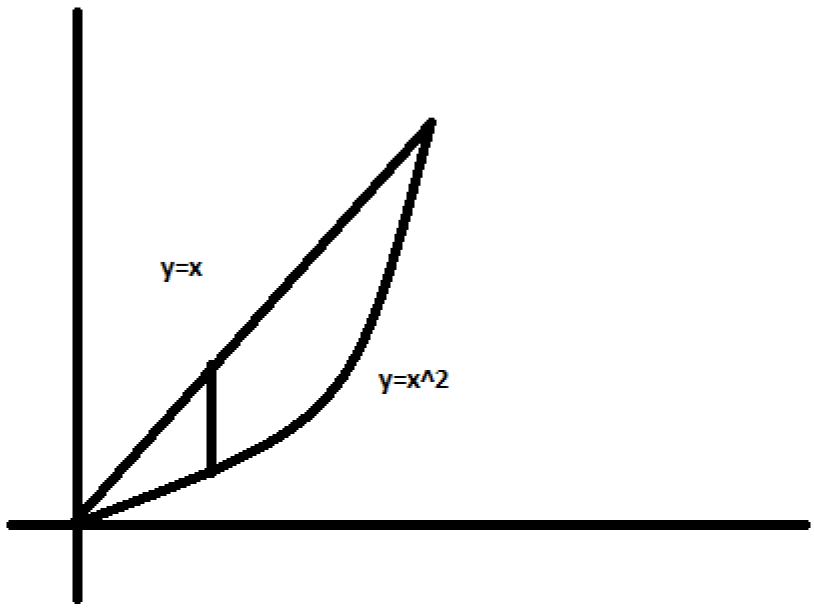
But with $y=x^3$ we have the radius $x=y^{1/3}$.

The area of the circles are then $A(y)=\pi x^2=\pi y^{2/3}$ so

$$V=\pi\int_0^8 y^{2/3}dy=\pi\left[\frac{y^{5/3}}{5/3}\right]_0^8=\frac{3\pi}{5}[32-0]=\frac{96\pi}{5}$$

Example 4: "Washer"

Let the area between $y = x$ and $y = x^2$ be spun around the x axis.
The area is now the area of an annulus or a ring, sometimes known as a washer.



The area function is $A(x) = \pi x^2 - \pi\left(x^2\right)^2$

Setting the two functions equal we find the points of intersection:

$x = x^2$

$x^2 - x = 0$

$x(x-1) = 0$

So we want to integrate as follows:

$$V = \pi\int_0^1 \left(x^2 - x^4\right)dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$$

Hand out Worksheet,

If time permits go over review sheet.