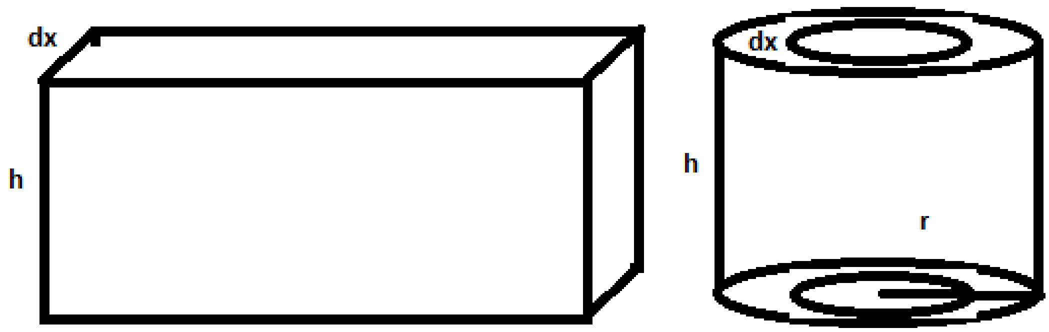
Lesson Plan 8 - 6.5 Volumes by Shells

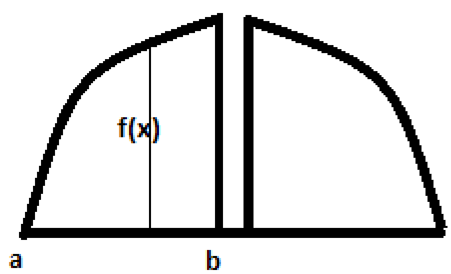
1) Take attendance

We start with the idea that you can take a rectangle with thin thickness and connect it into a cylinder.



The volume of the cylinder is approximately $2\pi rh(\triangle x)$

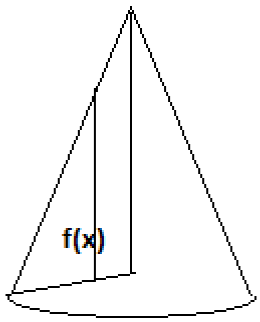
We conclude that if we have a more complex volume whose height is a function of the radius x:



$h = f(x)$

Then the volume is $V = \int_a^b 2\pi x f(x)\, dx$

Example 1: Volume of a cone



Note that if the height of the cone is h and the radius of the base is r

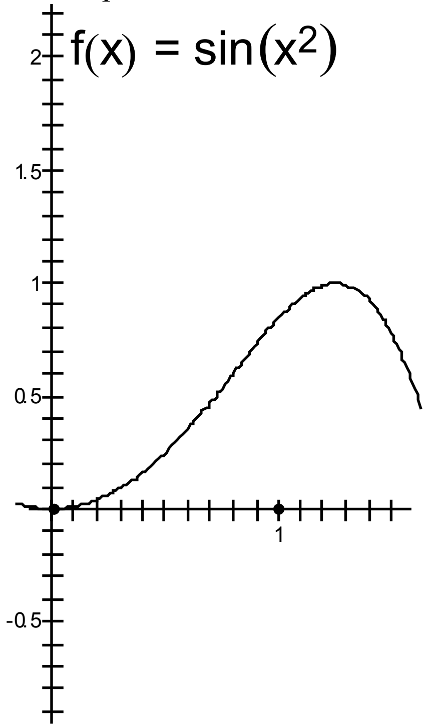
$$f\left(x\right)=h-\frac{h}{r}x$$

plugging this into our formula we get

$$V=\int_0^r 2\pi x\left(h-\frac{h}{r}x\right)dx=2\pi h\int_0^r x-\frac{x^2}{r}dx=$$

$$2\pi h\left[\frac{x^2}{2}-\frac{x^3}{3r}\right]_0^r=2\pi h\left[\frac{r^2}{2}-\frac{r^2}{3}-\left(0\right)\right]=\frac{2\pi hr^2}{6}=\frac{1}{3}\pi r^2 h$$

Example 2: A sine Bowl



Take the volume of the above function on the interval $\left[0, \sqrt{\pi/2}\right]$

Then we have

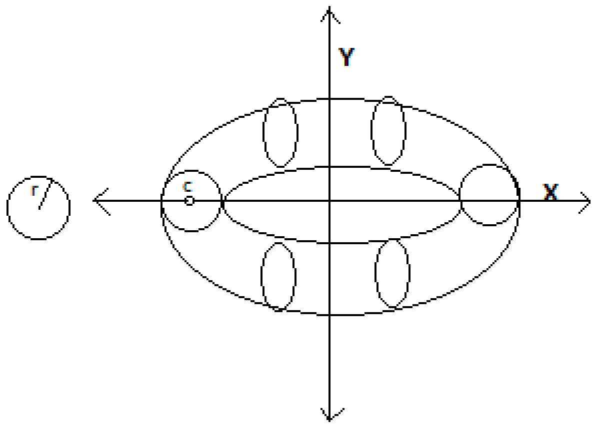
$$V = 2\pi \int_0^{\sqrt{\pi/2}} x \sin x^2 \, dx$$

Substitute

$$u = x^2$$
$$du = 2x \, dx$$

$$2\pi \int_0^{\sqrt{\pi/2}} x \sin x^2 \, dx = \pi \int_0^{\pi/2} \sin u \, du = \pi \left[-\cos x\right]_0^{\pi/2} = \pi\left(0 - {}^{-}1\right) = \pi$$

Example 3: Volume of a ring



Rotate a circle given by $(x-c)^2 + y^2 = r^2$ forming a ring, what is the volume

Note that the height of the circle at any point is

$f(x) = 2\sqrt{r^2 - (x-c)^2}$ on the interval $[c-r, c+r]$

so

$$V = \int_{c-r}^{c+r} 2\pi x 2\sqrt{r^2 - (x-c)^2}\, dx$$

Note that

$$\int_{c-r}^{c+r} 2x\sqrt{r^2 - (x-c)^2}\, dx = \int_{c-r}^{c+r} 2(x-c)\sqrt{r^2 - (x-c)^2}\, dx + \int_{c-r}^{c+r} 2c\sqrt{r^2 - (x-c)^2}\, dx$$

Let's do a transformation on the left term

$u = x - c$

$du = dx$

$$\int_{c-r}^{c+r} 2(x-c)\sqrt{r^2 - (x-c)^2}\, dx = \int_{-r}^{+r} 2u\sqrt{r^2 - u^2}\, dx$$

Since $u\sqrt{r^2 - u^2}$ is an odd function, the integral is zero.

$$V = (2\pi c)\left(2\int_{c-r}^{c+r} \pi\sqrt{r^2 - (x-c)^2}\,dx\right)$$

Doing the same transformation on

$$2\int_{c-r}^{c+r} \pi\sqrt{r^2 - (x-c)^2}\,dx = 2\int_{-r}^{+r} \pi\sqrt{r^2 - u^2}\,du$$

But this is just the area of a circle with radius r.

So finally we get

$$V = (2\pi c)(\pi r^2) = 2\pi^2 c r^2$$