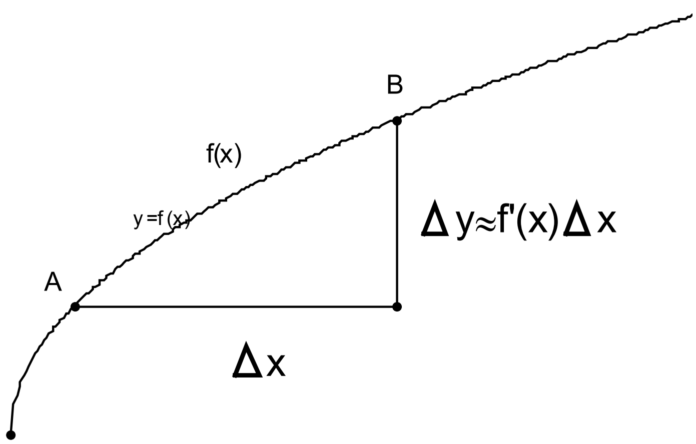
Lesson Plan 9B - 6.5 Length of Curves

1) Take attendance
2) Return Homework and Quiz
3) Questions on Quiz
4) MidTerm Thursday will include everything up through today and anything specified on Tuesday
5) Length of Curves

Assume we have a curve described by parametric functions $x = f(t)$ and $y = g(t)$ defined on some interval $a \le t \le b$.

As an approximation we can break the curve up as follows:



We know that $\overline{AB} \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2 + (f'(A)\Delta x)^2} = \sqrt{1 + (f'(A))^2}\,\Delta x$

If we have points $\{A = (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) = B\}$ with $\Delta x_i = x_{i+1} - x_i$

We can sum over small segments of the curve:

$$L \approx \sum_{i=1}^{n} \sqrt{1 + f'(x_i)^2}\,\Delta x_i$$

But then

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + f'(x_i)^2}\,\Delta x_i = \int_A^B \sqrt{1 + f'(x_i)^2}\,dx \text{ or}$$

$$L = \int_A^B \sqrt{1 + \left(\frac{df}{dx}\right)^2}\,dx$$

Example 1:

Find the length of the arch of the parabola $f(x) = x^{3/2}$ from $(0,0)$ to $(4,8)$

Here we treat y as the parameter so we have

$$L = \int_0^1 \sqrt{1+\left(\frac{df}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1+\frac{9}{4}x}\, dx =$$

Substitute $u = 1+\frac{9}{4}x$ so that $du = \frac{9}{4}dx$ or $dx = \frac{4}{9}du$

$$L = \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx = \frac{4}{9}\int_1^{10} \sqrt{u} = \frac{4}{9}\left[\frac{2u^{3/2}}{3}\right]_1^{10} = \frac{8}{27}\left[10^{3/2}-1\right]$$

Example 3:

What is the arc length of the curve $f(x) = 2e^x + \frac{1}{8}e^{-x}$ on the interval $[0, \ln 2]$

First we find $f'(x) = 2e^x - \frac{1}{8}e^{-x}$

and

$$[f'(x)]^2 = 4e^{2x} - \frac{1}{2} + \frac{1}{64}e^{-2x}$$

so

$$L = \int_0^{\ln 2} \sqrt{1 + 4e^{2x} - \frac{1}{2} + \frac{1}{64}e^{-2x}}\,dx = \int_0^{\ln 2} \sqrt{4e^{2x} + \frac{1}{2} + \frac{1}{64}e^{-2x}}\,dx = \int_0^{\ln 2} \sqrt{\left(2e^x + \frac{1}{8}e^{-x}\right)^2}\,dx =$$

$$\int_0^{\ln 2} 2e^x + \frac{1}{8}e^{-x}dx = \left[2e^x - \frac{1}{8}e^{-x}\right]_0^{\ln 2} = 4 - \frac{1}{16} - \left(2 - \frac{1}{8}\right) = 2 + \frac{1}{16} = \frac{33}{16}$$

Example: 2 Reversing the axis!

Find the length of the curve $f(x)=\ln\left(x+\sqrt{x^2-1}\right)$ on the interval $\left[1,\sqrt{2}\right]$

Note that the derivative of $f(x)$ is $\dfrac{1+\dfrac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}=\dfrac{x+\sqrt{x^2-1}}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}}=\dfrac{1}{\sqrt{x^2-1}}$

But this is undefined at $x=1$

So we reverse the role of *x* and *y*.
First find the new limits

$$\ln\left(1+\sqrt{1-1}\right)=\ln(1)=0$$

$$\ln\left(\sqrt{2}+\sqrt{\sqrt{2}^2-1}\right)=\ln\left(\sqrt{2}+1\right)$$

Now find $f'(x)$

$$x=\ln\left(y+\sqrt{y^2-1}\right)\to e^x=y+\sqrt{y^2-1}\to e^x-y=\sqrt{y^2-1}\to$$

$$\left(e^x-y\right)^2=y^2-1\to e^{2x}-2ye^x-y^2=y^2-1\to e^{2x}-2ye^x=-1\to$$

$$y=\frac{e^{2x}+1}{2e^x}=\frac{e^x+e^{-x}}{2}$$

So the length is

$$\int_0^{\ln\left(\sqrt{2}+1\right)}\sqrt{1+\left[\left(\frac{e^x+e^{-x}}{2}\right)'\right]^2}\,dx=\int_0^{\ln\left(\sqrt{2}+1\right)}\sqrt{1+\left[\left(\frac{e^x-e^{-x}}{2}\right)\right]^2}\,dx=$$

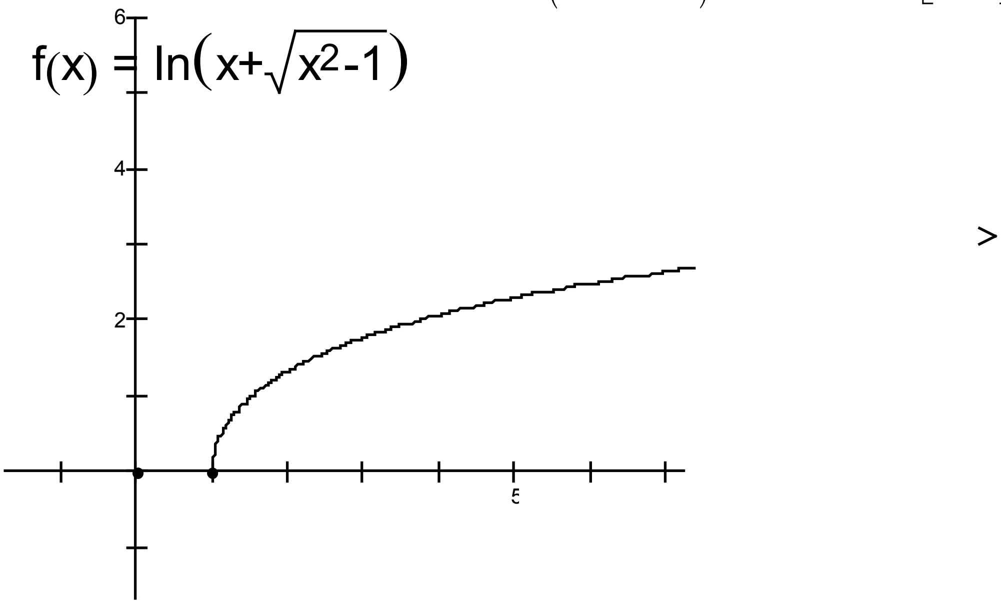
$$\int_0^{\ln\left(\sqrt{2}+1\right)}\sqrt{1+\left(\frac{e^{2x}-2+e^{-2x}}{4}\right)}\,dx=\int_0^{\ln\left(\sqrt{2}+1\right)}\sqrt{\frac{e^{2x}+2+e^{-2x}}{4}}\,dx=$$

$$\int_0^{\ln\left(\sqrt{2}+1\right)}\sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2}\,dx=\int_0^{\ln\left(\sqrt{2}+1\right)}\frac{e^x+e^{-x}}{2}\,dx=\left[\frac{e^x-e^{-x}}{2}\right]_0^{ln\left(\sqrt{2}+1\right)}=\frac{\sqrt{2}+1-\dfrac{1}{\sqrt{2}+1}}{2}=$$

$$\frac{2+2\sqrt{2}+1-1}{2\left(\sqrt{2}+1\right)}=\frac{2\left(\sqrt{2}+1\right)}{2\left(\sqrt{2}+1\right)}=1$$

Example 2: (Same Problem)

Find the length of the curve $y = f(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ on the interval $\left[1, \sqrt{2}\right]$



The problem with this function is that at 1, the tangent is vertical and therefore $f'(x)$ is undefined.

We note however that $f(x)$ is actually the $\cosh^{-1}(x) = \dfrac{e^x + e^{-x}}{2}$

Proof:

$y = \ln\left(x + \sqrt{x^2 - 1}\right)$

$e^y = x + \sqrt{x^2 - 1}$

$e^y - x = \sqrt{x^2 - 1}$

squaring both sides gives us

$e^{2y} - 2xe^y + x^2 = x^2 - 1$

$e^{2y} - 2xe^y = -1$

$2xe^y = e^{2y} + 1$

$x = \dfrac{e^y + e^{-y}}{2} = \cosh(y)$

If $x \in \left[1, \sqrt{2}\right]$ then $y \in \left[0, \ln\left(\sqrt{2} + 1\right)\right]$

So

$$L = \int_0^{\ln(\sqrt{2}+1)} \sqrt{1-[\cosh' y]^2}\,dy = \int_0^{\ln(\sqrt{2}+1)} \sqrt{1-(\sinh y)^2}\,dy = \int_0^{\ln(\sqrt{2}+1)} \sqrt{(\cosh y)^2}\,dy = \int_0^{\ln(\sqrt{2}+1)} \cosh y\; dy =$$

$$[\sinh y]_0^{\ln(\sqrt{2}+1)} = \sinh\left(\ln\left(\sqrt{2}+1\right)\right) - \sinh(0) = \frac{e^{l\,n(\sqrt{2}+1)} - e^{-l\,n(\sqrt{2}+1)}}{2} - \frac{e^0 - e^{-0}}{2} =$$

$$\frac{\sqrt{2}+1-1/\sqrt{2}+1}{2} = \frac{2+2\sqrt{2}+1-1}{2\left(\sqrt{2}+1\right)} = \frac{1+\sqrt{2}}{1+\sqrt{2}} = 1$$

Try Handout Problems