Lesson Plan 10 - Improper Integrals Part 2 5.10

1) Take attendance
2) Return Quiz, questions

Improper Integrals Part 2

Continuing on about Improper integrals:

Example 3: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Since both limits are infinite, we need to break this up into two parts:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

Each of these is evaluated as a separate limit:

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_{0}^{t} \frac{1}{1+x^2}\,dx =$$

$$\lim_{t\to-\infty}\left[\tan^{-1} x\right]_t^0 + \lim_{t\to\infty}\left[\tan^{-1} x\right]_0^t = \lim_{t\to-\infty}\left[0-\tan^{-1} t\right] + \lim_{t\to\infty}\left[\tan^{-1} t - 0\right] = \frac{\pi}{2}+\frac{\pi}{2}$$

Example 4: Now we ask for what values of p is the integral

$\int_{1}^{\infty} \frac{1}{x^p}\,dx$ Convergent?

$$\int_{1}^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_{1}^{t} \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[-\frac{1}{(p-1)x^{p-1}}\right]_1^t = \frac{1}{1-p}\lim_{t\to\infty}\left[\frac{1}{t^{p-1}}-1\right]$$

Note that if $p > 1$ then p-1 > 0 and $\lim_{t\to\infty}\frac{1}{t^{p-1}} = 0$

if $p \le 1$ then $\lim_{t\to\infty}\frac{1}{t^{p-1}} \to \infty$ so the integral is divergent.

Discontinuous Integrals, Improper Integrals Type 2

Example 5: $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

Note that the function has an asymptote as x=2. So we define the integral as follows:

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2}\int_t^5 \frac{1}{\sqrt{x-2}} = \lim_{t\to 2}\left[2\sqrt{x-2}\,\right]_t^5 = \lim_{t\to 2}\left[2\sqrt{3}-2\sqrt{t-2}\,\right] = 2\sqrt{3}$$

If the discontinuity is in the middle of the interval, the integral must be broken up into two parts as follows:

Example 7:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1} = \lim_{t\to 1}\int_0^t \frac{dx}{x-1} + \lim_{t\to 1}\int_t^3 \frac{dx}{x-1} =$$

$$\lim_{t\to 1}\left[\ln|x-1|\right]_0^t + \lim_{t\to 1}\left[\ln|x-1|\right]_t^3 = \lim_{t\to 1}\left[\ln|t-1| - \ln 1\right] + \lim_{t\to 1}\left[\ln|2| - \ln|t|\right]$$

But note that $\lim_{t\to 1}\ln|x-1|$ is divergent, so the integral is divergent.

Warning: Notice that if you don't divide the integral up properly you get a false answer:

$$\int_0^3 \frac{dx}{x-1} = \left[\ln|x-1|\right]_0^3 = \ln 2 - \ln 1 = \ln 2$$

Example 8:

$$\int_0^1 \ln x\,dx = \lim_{t\to 0}\int_t^1 \ln x\,dx$$

To find the anti-derivative we use integration by parts:

$$f = \ln x \quad g' = 1$$

$$f' = \frac{1}{x} \quad g = x$$

$$\lim_{t\to 0}\int_t^1 \ln x\,dx = \lim_{t\to 0}\left[x\ln x - \int 1\,dx\right]_t^1 = \lim_{t\to 0}\left[x\ln x - x\right]_t^1 = \lim_{t\to 0}\left[\ln 1 - 1 - (t\ln t - t)\right] =$$

$$\lim_{t\to 0}\left[-t\ln t - 1 + t\right]$$

The 2nd two terms have a limit of -1 but the first part requires L'Hospital's rule again.

$$\lim_{t\to 0} t\ln t = \lim_{t\to 0}\frac{\ln t}{1/t} = \lim_{t\to 0}\frac{\frac{d}{dt}\ln t}{\frac{d}{dt}1/t} = \lim_{t\to 0}\frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to 0} -t$$

So we have $\int_0^1 \ln x\, dx = -1$

Stop and do some problems from the handout.

Comparison tests with Improper integrals.

Sometimes it is important to know if an integral will converge or not, even though the function being integrated may not have an anti-derivative. An example is

$$\int_0^\infty e^{-x^2}\, dx$$

In circumstances like this, we can sometimes use the comparison test to get an answer.

The Comparison test.

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for all $x \geq a$

Then if $\int_a^\infty f(x)\, dx$ is convergent, then $\int_a^\infty g(x)\, dx$ is convergent.

Also if $\int_a^\infty g(x)\, dx$ is divergent, then $\int_a^\infty f(x)\, dx$ is divergent.

It makes sense if you see this in terms of areas under a curve. A diagram here might help.

Example 9:

Show that $\int_1^\infty e^{-x^2}dx$ is convergent.

Note that for all $x \geq 1$ that $e^{-x} \geq e^{-x^2} \geq 0$

Also note that $\int_1^\infty e^{-x}dx = \lim_{t\to\infty}\int_1^t e^{-x}dx = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t = \lim_{t\to\infty}\left[-e^{-t}+e^{-1}\right] = \frac{1}{e}$

This proves that $\int_1^\infty e^{-x^2}dx$ is convergent.

Sometimes this works by comparing to a simpler function:

Example:

Show that $\int_0^\infty \frac{\sin x}{e^x}dx$ is convergent.

Note that for $x \geq 0$ that $\frac{\sin x}{e^x} \leq \frac{1}{e^x}$

But since $\int_0^\infty \frac{1}{e^x}dx$ is convergent, then $\int_0^\infty \frac{\sin x}{e^x}dx$ must be convergent.

Homework assignment.