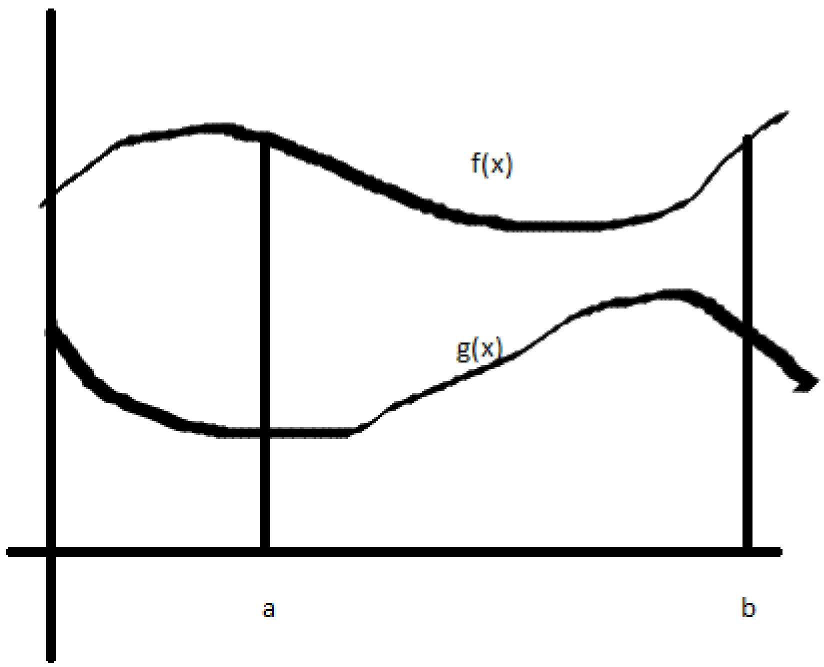
Lesson Plan 11 - Areas 6.1

1) Take attendance

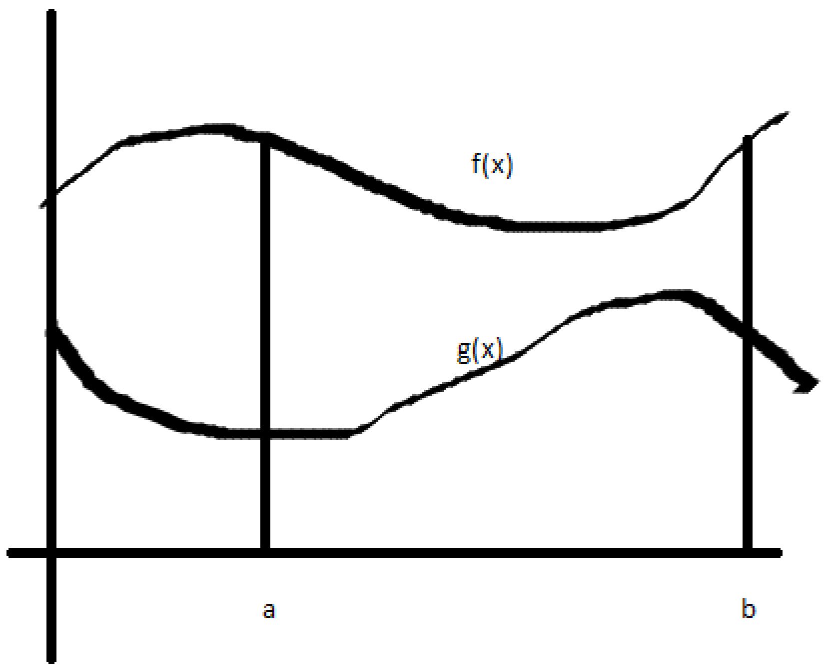
We've been looking at a definite integral as the area beneath a curve, that is the area between the curve and y=0.

If the y coordinate of the curve is < 0 we treat this as negative area.
What about the area between two curves?



Clearly the area below f(x) minus the area below g(x) is the area between the curves.

What if one or both functions drop below the X axis?

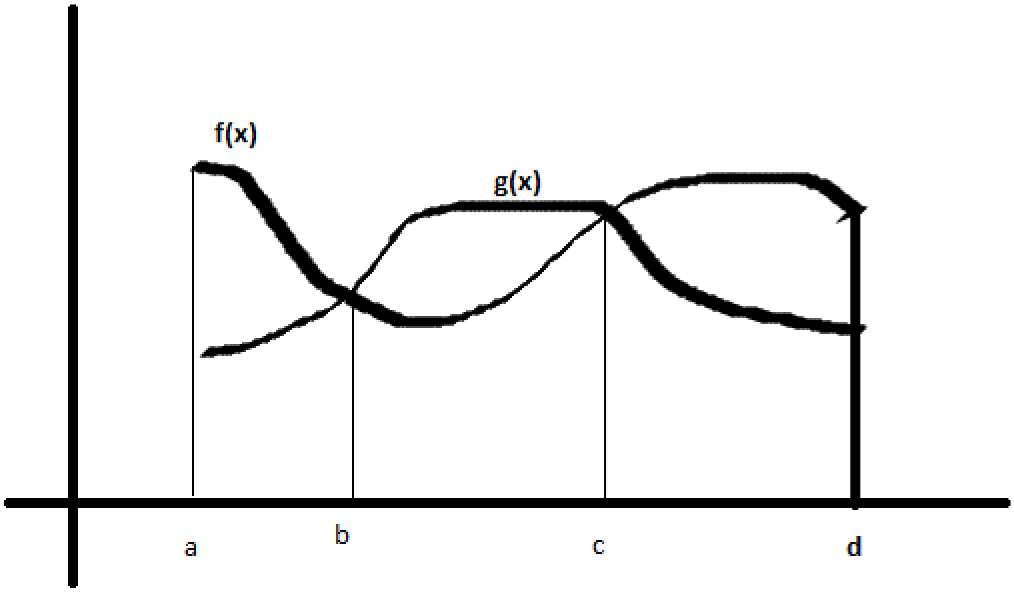


We can add a constant amount to both functions, moving them up above the line preserving the area. Then:

$$\int_a^b f(x)dx + C - \left[\int_a^b g(x)dx + C\right] = \int_a^b f(x)dx - \int_a^b g(x)dx + \int_a^b C\,dx - \int_a^b C\,dx =$$

$$\int_a^b f(x)dx - \int_a^b g(x)dx$$

What if we have two functions that cross over and we want all the area between them?



Then we need to calculate

$$\int_a^d |f(x)-g(x)|\,dx = \int_a^b f(x)-g(x)dx - \int_b^c f(x)-g(x)dx + \int_c^d f(x)-g(x)dx \int_a^b$$

Example 2: Find the area enclosed by $y = x^2$ and $y = 2x - x^2$

Setting these equal we find

$$x^2 = 2x - x^2$$
$$2x^2 - 2x = 0$$
$$x(x-1) = 0$$

So the points of intersection are 0 and 1.

We integrate $\int_0^1 \left|2x - x^2 - \left(x^2\right)\right| = \int_0^1 \left|2x - 2x^2\right| = \left|\left[x^2 - \frac{2x^3}{3}\right]_0^1\right| = \left|1 - \frac{2}{3} - (0-0)\right| = \frac{1}{3}$

Example 3: Find the area bounded by $y = e^x$ and $y = x$ on the interval [0,1]

Since $e^x > x$ for all $x \in [0,1]$

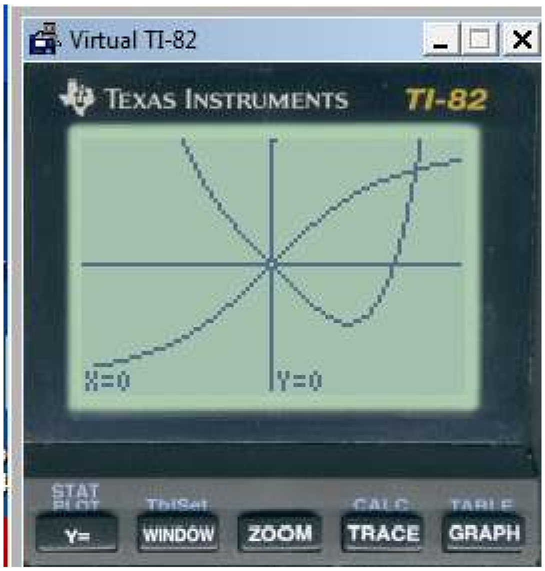
$$A = \int_0^1 \left(e^x - x\right) dx = \left[e^x - \frac{x^2}{2}\right]_0^1 = e - \frac{1}{2} - (1-0) = e - \frac{3}{2}$$

Example 3:

Sometimes it's difficult or impossible to find the points of intersection between curves, but we can approximate them with the calculator:

$$y = \frac{x}{\sqrt{x^2+1}} \qquad y = x^4 - x$$

Clearly (0,0) is a point of insersection.



Using the calculator's Calc/5:Intersection key we find the other intersection at 1.18

$$\int_0^{1.18} \frac{x}{\sqrt{x^2+1}} - \left(x^4 - x\right) dx$$

We could solve this using anti-derivatives, but since it is already an approximation, use the Calc/7:Integration function giving .78538855

Example 4: from the book, We have the speed of two cars at equal intervals

t	0	2	4	6	8	10	12	14	16
v_A	0	34	54	67	76	84	89	92	95
v_B	0	21	34	44	51	56	60	63	65
$v_A - v_B$	0	13	20	23	25	28	29	29	30

How can we calculate the integral over $v_A - v_B$?

Simpson's Rule?

What does this "Area" that we are calculating represent?

Sometimes it is easier to integrate along Y instead of X?

Example 5: Find the area between $y = x - 1$ and $y^2 = 2x + 6$

Already we have a problem because the 2nd equation is not a function.

But we can switch x and y

$x = y - 1$ and $x^2 = 2y + 6$ or

$y = x + 1$ and $y = \frac{x^2 - 6}{2}$

We find the intersection $x + 1 = \frac{x^2 - 6}{2}$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

So

$$\int_{-2}^{4} x + 1 - \left(\frac{x^2 - 6}{2}\right) dx = \int_{-2}^{4} -\frac{x^2}{2} + x + 4\,dx = \left[-\frac{x^3}{6} + \frac{x^2}{2} + 4x\right]_{-2}^{4} =$$

$$-\frac{64}{6} + 8 + 16 - \left(\frac{-8}{6} + 2 - 8\right) = 18$$

Integrating using parametric equations.

Let's say we have a function F(x) which we can't be described in terms of x but we have parametric equations.

$x = g(t) \qquad y = f(t)$

so $y = F(g(t)) = f(t)$

The substitution rule tells us

$$\int_{\alpha}^{\beta} F(g(t))g'(t)\,dt = \int_{g(\alpha)}^{g(\beta)} F(x)\,dx$$

Example 6: The area under one cycle of a cycloid, $2\pi r$.

First Use calculator to show that for parametric equations
$x = \theta - \sin\theta$ and $y = 1 - \cos\theta$ over $[0, 2\pi]$ that the integral is 3π.

$x = g(\theta) = r(\theta - \sin\theta) \qquad y = f(\theta) = r(1 - \cos\theta)$

At $\theta = 0$ and $\theta = 2\pi$ we have $(0,0)$ and $(2\pi r, 0)$

$$\int_0^{2\pi r} F(x)\,dx = \int_0^{2\pi} f(\theta)g'(\theta)\,d\theta = \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)\,d\theta =$$

so $r^2 \int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\,d\theta = r^2 \int_0^{2\pi} 1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\,d\theta =$

$$r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = r^2\left[\frac{3}{2}2\pi - 0 + 0 - (0 - 0 + 0)\right] = 3\pi r^2$$