Lesson Plan 19 - Separable Variables, Linear Differential Equations

1) Take attendance
2) Announce Quiz next Tuesday, last quiz
3) Homework questions?

Separable Fields

One kind of differential equation that lends itself to a direct solution is one with separable variables.

if we have $\frac{dy}{dx} = g(x)f(y)$ where g and f only involve x and y respectively then we can perform the following hand-waved approach:

$$\frac{1}{f(y)}dy = g(x)dx$$

Note this is really notational. dy and dx are not elements that can be divided up this way, however the usefulness of this strategy shows some of the advantages of the Leibniz notation over Newtonian.

From here we proceed to $\int \frac{1}{f(y)}\,dy = \int g(x)\,dx$

Assuming we can evaluate both of these integrals, we will have an implicit solution, we may possibly be able to solve for y giving us an explicit solution.

We can verify that this hand-wave is correct by finding the derivative of both sides.

$$\frac{d}{dx}\int \frac{1}{f(y)}\,dy = \frac{d}{dx}\int g(x)\,dx$$

First we see that

$$\frac{d}{dx}\int g(x)\,dx = g(x)$$

Next using the chain rule we have

$$\frac{1}{f(y)}\frac{dy}{dx} = g(x) \rightarrow \frac{dy}{dx} g(x)f(y)$$

Putting these together we have

$$\frac{d}{dx}\int \frac{1}{f(y)}\,dy = \frac{d}{dy}\left(\int \frac{1}{f(y)}dy\right)\frac{dy}{dx} = \frac{1}{f(y)}\frac{dy}{dx}$$

Example:

$$\frac{dy}{dx}=\frac{x^2}{y^2}\to\int y^2dy=\int x^2dx\to\frac{y^3}{3}=\frac{x^3}{3}+C\to y=\sqrt[3]{x^3+3C}\to y=\sqrt[3]{x^3+D}$$

Given initial conditions $y(0)=1$ we have $\sqrt[3]{D}=1\to D=1$ for a solution of
$y=\sqrt[3]{x^3+1}$

Example: with an implicit solution

$$\frac{dy}{dx}=\frac{6x^2}{2y+\cos y}\to\int 2y+\cos y\;dy=\int 6x^2dx\to y^2+\sin y=2x^3+C$$

Here it is impossible to solve for *y,* however any function that fulfills this equation will be a solution to the differential equation.

Example:

$$\frac{dy}{dx}=x^2y$$

$$\int\frac{dy}{y}=\int x^2dx\to\ln|y|=\frac{x^3}{3}+C$$

We raise both sides to the power if *e*:

$$e^{\ln|y|}=e^Ce^{x^3/3}\to|y|=De^{x^3/3}\to y=\pm De^{x^3/3}$$

Note that $D=0\to y=0$ is also a solution, so we have

$$y=Ae^{x^3/3}\quad A\in\mathbb{R}$$

Distribute Handout

Linear Differential Equations

Consider a generic linear differential equation:

$$\sum_{i=0}^{n} a_i y^{(i)'} = 0 \text{ where } y^{(0)'} \equiv y$$

We can try as a possible solution $y = e^{rx}$.

So $y^{(i)'} = r^i e^{rx}$ and

$$\sum_{i=0}^{n} a_i y^{(i)'} = \sum_{i=0}^{n} a_i k^i e^{rx} = 0$$

Multiplying through by e^{-rx} we get

$$\sum_{i=0}^{n} a_i r^i = 0$$

This is an n'th degree polynomial equation which will have n roots, $\{r_1, r_2, \ldots, r_n\}$

So for each such root, $y = e^{r_i x}$ will be a solution, and therefore $\sum A_i e^{r_i x}$ will be a solution.

If n=1 we have the simple equation $y' + ay = 0$ with the solution $y = e^{-ax}$

For a 2nd order equation, $y'' + ay' + by = 0$

if $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$

So the equation becomes $r^2 e^{rx} + are^{rx} + be^{rx} = 0$

If we multiply through again by e^{-rx} we get

$r^2 + ar + b = 0$ a quadratic equation whose solutions are

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

This should look familiar. Looking at the discriminant

if $a^2 = 4b$ the solution simplifies to $y = e^{(-a/2)x}$.

If $a^2 > 4b$ then we get two solutions

with $r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}$ and $r_1 = \frac{-a - \sqrt{a^2 - 4b}}{2}$

we have $y = e^{r_1 x}$ and $y = e^{r_2 x}$ and so the general solution is

$$y = Ae^{r_1 x} + Be^{r_2 x}$$

Finally if $a^2 < 4b$ the roots are complex and we end up with solutions:

$$y = Ae^{-ax} \cos\beta + Be^{-ax} \sin\beta \text{ where } \beta = \frac{\sqrt{4b - a^2}}{2}$$

Example:

$y'' + 5y' + 4y = 0$ where $y(0) = 0$ and $y'(0) = 1$

Since $5^2 - 4 \cdot 4 = 9 > 0$ $r_1 = \frac{-5 + \sqrt{9}}{2} = -1$ and $r_2 = \frac{-5 - \sqrt{9}}{2} = -4$

so $y = Ae^{-x} + Be^{-4x}$ and $y' = -Ae^{-x} + -4Be^{-4x}$

The initial conditions tell us that $y(0) = 0 = A + B$ and $y'(0) = 1 = -A - 4B$

Solving these two equations we get $A = -\frac{1}{3}$ and $B = \frac{1}{3}$ so the final solution is

$$y = \frac{-e^{-x} + e^{-4x}}{3}$$