Lesson Plan 2 - First Day of Class

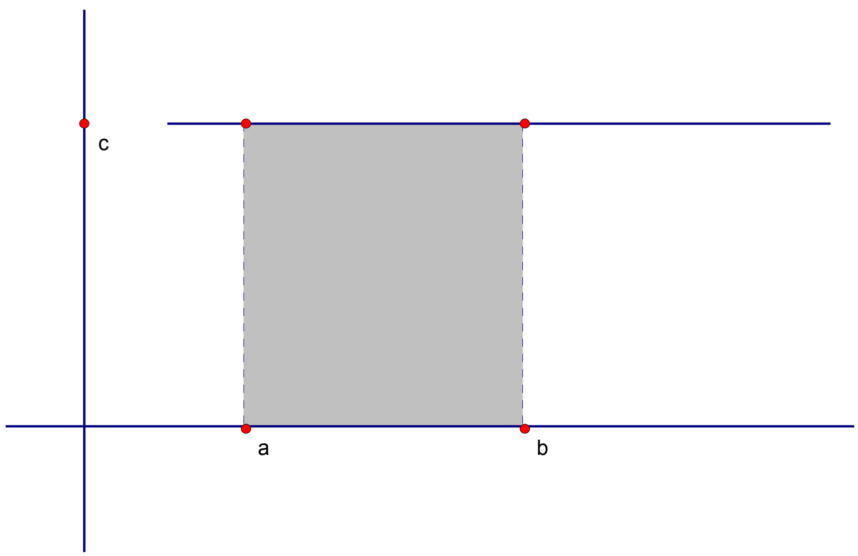
1) Take attendance, any new students?

2) Questions on the homework or the work sheet?

3) Go over problem 6 on the worksheet.
How to find the intersection and how to calculate the area with the calculator.

4) Some simple examples of calculating the area under a curve

$f(x) = c$

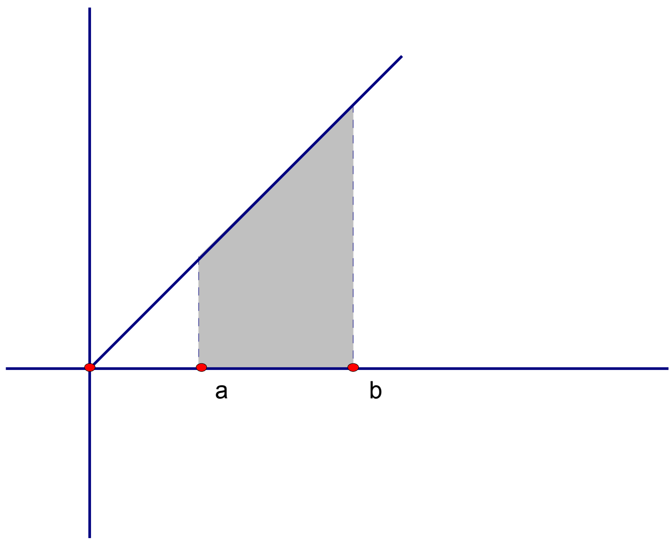


Note that the area under the curve is $A = (b-a)c$

We could write this as $xc\big|_a^b$ or $[xc]_a^b$ where the notation indicates you evaluate the xc at the upper limit and subtract the xc evaluated at the lower limit.

Note that we could also have written this as $[F(x)]_a^b$ where $F'(x) = f(x)$

We try this again for a slight more complex function $f(x) = x$



Here the area can be seen as the difference in area of the two triangles at points:
{(0,0), (a,0), (a,*f*(a))} and
{(0,0), (b,0), (b,*f*(b))}

$$A = \frac{b^2}{2} - \frac{a^2}{2} = \left[\frac{x^2}{2}\right]_a^b$$

Note that we could also have written this as $\left[F(x)\right]_a^b$ where $F'(x) = f(x)$

Some Review

We spoke on Tuesday of the definite integral, a set of symbols that indicate the area under a function.

$$\int_a^b f(x)\,dx$$

We now would like to investigate how we might calculate this function in a more direct and exact manner than before.

The preceding examples suggest that

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b \quad = F(b) - F(a)$$

$$Where\ F'(x) = f(x)$$

$$F(x)\ being\ the\ Anti-Derivative\ of\ f(x)$$

Is a possible solution.

Let's proceed by defining a function as follows:

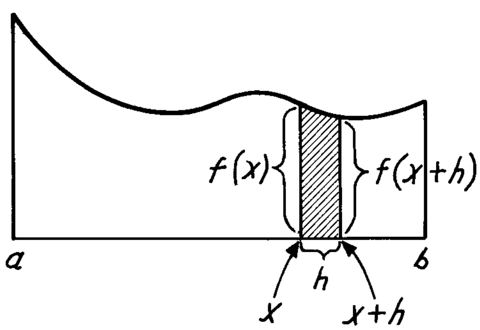
$$F(x) = \int_a^x f(t)\,dt$$

Notice that this is a function of *x* and not a definite integral.
It is a function which simply indicates the area under the curve *f*(*x*) from the point *a* the unknown point *x*.

Now consider this limit, which should look familiar:

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

What does that look like graphically?



$$\begin{aligned} F(x+h) &= \text{area from } a \text{ to } x+h \\ F(x) &= \text{area from } a \text{ to } x \\ F(x+h) - F(x) &= \text{area from } x \text{ to } x+h \\ \frac{F(x+h) - F(x)}{h} &= \frac{\text{area from } x \text{ to } x+h}{h} \overset{\text{approx}}{=} f(x) \text{ for small } h. \end{aligned}$$

In this diagram you can see that as $h \to \infty$ the shaded area comes closer and closer to being a rectangle with area $\frac{f(x+h) + f(x)}{2} h$

As such

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{f(x+h) + f(x)}{2} = f(x)$$

By the definition of the derivative, that means that

$$F'(x) = f(x)$$

That is *F*(*x*) is the anti-derivative of *f*(*x*)

Let's let that settle in a bit with a few examples:

What is the area beneath the function $y = x^2$ between 2 and 4?

$$\int_2^4 x^2 dx = \left[\frac{x^3}{3}\right]_2^4 = \frac{64}{3} - \frac{8}{3} = \frac{56}{3}$$

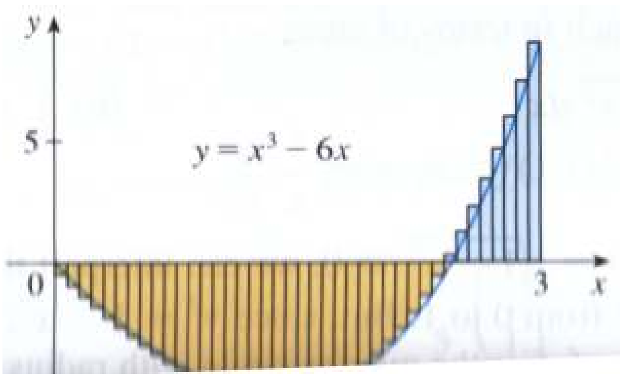
What is the area beneath the function $y = \cos(x)$ between $\frac{\pi}{3}$ and $\frac{\pi}{2}$?

$$\int_{\pi/3}^{\pi/2} \cos(x)\,dx = \left[\sin(x)\right]_{\pi/3}^{\pi/2} = \sin(\pi/2) - \sin(\pi/3) = \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} = \frac{\sqrt{2} - \sqrt{3}}{2}$$

What is the area beneath the function $y = e^x$ between 0 and 2?

$$\int_0^2 e^x dx = \left[e^x\right]_0^2 = e^2 - e^0 = e^2 - 1$$

Also note that if $f(x) < 0$, the area is negative:



Here the definite integral might be positive or negative depending on the limits.

Note that not all functions are integrable.
For example:

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{Q}^c \end{cases}$$

Condition for Integrablity:
If $f(x)$ is a continuous on $[a,b]$ or has at most a finite number of jump discontinuties then $f(x)$ is integrable on $[a,b]$,

that is $\int_a^b f(x)dx$ exists!

Other Properties of an Integral (that you should know)

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b f(x)-g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$\int_a^b f(x)+g(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

if $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)dx \geq 0$

if $f(x) \geq g(x)$ for $a \leq x \leq b$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

if $m \leq f(x) \leq M$ for $a \leq x \leq b$ then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$