Lesson Plan 20 - More Linear Differential Equations
and Orthogonal Trajectories

1) Take attendance
2) Return Homework
3) Announce Quiz next Tuesday, last quiz
4) Homework questions?

More on Linear Differential Equations

We saw on Tuesday that for a linear differential equation:

$$\sum_{i=0}^{n} a_i y^{'(i)} = 0$$

that there is an associated polynomial equation

$\sum_{i=0}^{n} a_i r^i = 0$ whose roots $r = \{r_1, r_2, \ldots, r_n\}$ have the property that

$y = e^{r_i x}$ is a solution and therefore $y = \sum_{i=1}^{n} A_i e^{r_i x}$ is a solution.

A question was left over from Tuesday as to what happens when there are duplicate roots.

I suggested that for an equation such as:

$y'' - 2y' + y = 0$ that $y = xe^{rx}$ might also be a solution.   Let's see:

The associated polynomial here is $r^2 - 2r + 1 = 0 \rightarrow (r-1)^2 = 0$
so r=1, but it is a duplicated root.

We know that $y = e^x$ must be a solution, but let's check $y = xe^x$

$$y' = xe^x + e^x$$
$$y'' = xe^x + 2e^x$$
$$y'' - 2y' + y = xe^x + 2e^x - 2\left[xe^x + e^x\right] + xe^x = xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0$$

This is highly suggestive that when a differential equation's associated polynomial has n duplicated roots that solutions:

$y = x^i e^{rx}$ where $i \in \{0, 1, \ldots n\}$

This suggests something quite beautiful about these equations, but to explain this, I will have to digress and talk about a pre-calculus subject, Vectors.

You may recall that a vector is a directed line segment with a head and a tail.
We add vectors by positioning the tail of one vector at the head of the second, with the sum being the vector starting at the tail of the first vector and ending at the tail of the 2nd vector. So we have the property that if

1) $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are vectors, then $\overrightarrow{v_1}+\overrightarrow{v_2}$ is a vector.

2) Also we have that if $a \in \mathbb{R}$ and $\vec{v}$ is a vector then $\overrightarrow{av}$ is a vector.

3) Finally there is a special vector called the zero vector which is written $\vec{0}$ which has the property that if $\vec{v}$ is a vector then $\vec{v}+\vec{0}=\vec{v}$.

When an objects such as a 3D Euclidean vectors have these three properties they are called a VECTOR SPACE.

There are other vector spaces besides 3D Euclidean vectors.

Before we go on, there are a few properties of a vector space that we need to explore.

In the case of the 3D Euclidean vector space we have defined three vectors $\vec{i},\vec{j},\vec{k}$ which are unit vectors in the *x,y* and *z* directions respectively. They have the property that any vector can be expressed as

$$\vec{V}=a\vec{i}+b\vec{j}+c\vec{k}$$

A property of all vector spaces is that they will have a set of vectors such as these that is as small as possible and that can be used to generate any vector in the space. Such as set is called a BASIS. A basis is not unique. Consider the set $\left\{2\vec{i},2\vec{j},2\vec{k}\right\}$ which is a basis for the same vector space as $\left\{\vec{i},\vec{j},\vec{k}\right\}$.

The property that all bases of a vector space have in common is that they have the same number of elements. This number is called the DIMENSION of the vector space.

The meaning of dimension might seem obvious in a 3D Euclidean vector space, however it is not quite so obvious in other contexts.

Back to linear differential equations

It would seem then that the solutions to the linear differential equations form a vector space. Let's check the properties.

If $f(x)$ and $g(x)$ are solutions to the differential equation $\sum_{i=0}^{n} a_i y^{'(i)} = 0$ then we have

$\sum_{i=0}^{n} a_i f(x)^{'(i)} = 0$ and $\sum_{i=0}^{n} a_i g(x)^{'(i)} = 0$ adding these together we have

$$\sum_{i=0}^{n} a_i f(x)^{'(i)} + \sum_{i=0}^{n} a_i g(x)^{'(i)} = \sum_{i=0}^{n} a_i \left(f(x)^{'(i)} + g(x)^{'(i)} \right) = 0$$

This shows the first property.

If $f(x)$ is a solution to $\sum_{i=0}^{n} a_i y^{'(i)} = 0$ and $b \in \mathbb{R}$ then

$b\sum_{i=0}^{n} a_i f(x)^{'(i)} = \sum_{i=0}^{n} a_i bf(x)^{'(i)} = 0$ and so $bf(x)$ is a solution.

Finally if $y = 0$ then all derivatives of y are zero and so $\sum_{i=0}^{n} a_i y^{'(i)} = 0$.

So y=0 is the zero vector in this vector space.

It should now be clear how to construct a basis for such a vector space.

1) Find the roots of the associated polynomial.
By the fundamental theorem of Algebra we know there will be n where n is the degree of the associated polynomial as well as the order of the differential equation.
2) If a root r is singular, then $x = e^{rx}$ is a solution and a basis member.
3) If a root r is non-singular, duplicated n times, then $\left\{ e^{rx}, xe^{rx}, \cdots, x^n e^{rx} \right\}$ are all solutions and basis members.

So finally we can conclude that the solutions of an n^{th} order linear differential equation form a vector space.

Pass out handout

Go over

Break

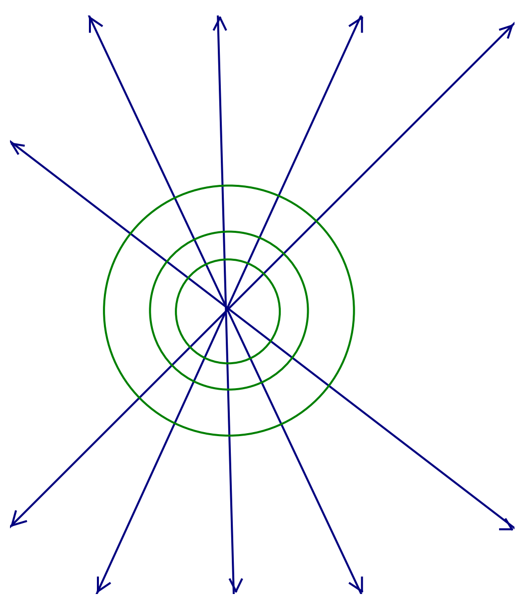
Orthogonal Trajectories

First off, please give me a definition of ORTHOGONAL?

An orthogonal trajectory is a family of curves that intersect each curve of another family orthogonally. that is at right angles.

The simple example would be $y=b$ is an orthogonal trajectory for $x=a$.

Another simple example would be $x^2 + y^2 = R^2$, the set of circles centered at (0,0) and $y = ax$ the set of lines through the origin.



Example:

What about the family of curves $x = ky^2 \to \frac{dy}{dx} = \frac{y^2}{2xy} = \frac{y}{2x}$.

What does this family look like?

To find an orthogonal trajectory for this family, we first find a differential equation that is satisfied by all members of the family.

First we differential implicitly.

$$\frac{d}{dx}x = \frac{d}{dx}ky^2 \to 1 = 2ky\frac{dy}{dx} \text{ so } \frac{dy}{dx} = \frac{1}{2ky}.$$

To eliminate the k we take the original equation and solve for k and then substitute.

$$k = \frac{x}{y^2} \to \frac{dy}{dx} = \frac{y^2}{2xy} = \frac{y}{2x}$$

To be an orthogonal trajectory the slopes of the curves must be the negative reciprocal so it must be a solution to

$$\frac{dy}{dx} = -\frac{2x}{y}$$

We solve this by separating the variables

$$\int y\,dy = -\int 2x\,dx \to \frac{y^2}{2} = -x^2 + C \to \frac{y^2}{2C} + \frac{x^2}{C} = 1$$

This is the equation of an ellipse with major axis C along y and minor axis $C/2$ along the x axis.

Let's try some examples from the handout.

Go over