Lesson Plan 4 - The Fundamental Theorem of Calculus 5.4

1) Take attendance
2) Quiz next Tuesday
3) Questions on the homework or the work sheet?

We have previously discussed the very important connection between Differential and Integral Calculus known as the Fundamental Theorem of Calculus.

In fact, calculating definite integrals and evaluating indefinite integrals would be very difficult without this connection.

There are a number of forms of this theorem but one version divides it into two parts:

For both parts we assume that *f* is continuous on [a,b].

Part 1

$\left[\int_a^x f(x)\,dx\right]' = f(x)$ or written in Leibniz notation

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Part 2

If $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$

A note about combining the chain rule with the fundamental theorem
(Example 5, page 370)

$\frac{d}{dx}\int_a^{x^4} \sin(t)\,dt$ In this case we must be careful since our endpoint is a function of the independent variable. We must invoke the chain rule as follows:

First set $u = x^4$

Then our integral becomes $\frac{d}{dx}\left[\int_a^u \sin(t)\,dt\right]\frac{du}{dx} = \sin(u)\frac{du}{dx} = \sin\left(x^4\right)\cdot 4x^3$

Give Students Class Handout 4 to work on.
Demarcation - The material required for the Quiz stops here

The substitution Rule 5.5

This tool can help us evaluate some complicated integrals by making them simpler.
Applying the rule is not straight forward.
It takes some guess work, some intuition and therefore some practice.

The substitution rule is inspired by working the chain rule backwards.

$$\left[f\left(g\left(x\right)\right)\right]' = f'\left(g\left(x\right)\right)g'\left(x\right)$$

So we can see that for

$$\int f'\left(g\left(x\right)\right)g'\left(x\right)dx$$

if we can pick

$$u = g\left(x\right)$$

therefore

$$\frac{du}{dx} = g'\left(x\right)$$

or tweaking the notation

$$du = g'\left(x\right)dx$$

Then $\int f'\left(g\left(x\right)\right)g'\left(x\right)dx = \int f'\left(u\right)du = f\left(u\right) + C$

Example 1:

$$\int \frac{dx}{1+\sqrt{x}}$$

we set $x = u^2$

so $\frac{dx}{du} = 2u$ or $dx = 2u\,du$

Then $\int \frac{dx}{1+\sqrt{x}} = \int \frac{2u\,du}{1+u} = \int \frac{2\left(1+u\right)-2}{1+u}du = \int 2 - \frac{2}{1+u}du = 2u - 2\ln\left|1+u\right|$

Now we substitute back u

$$2u - \ln\left|1+u\right| = 2\sqrt{x} - 2\ln\left|1+\sqrt{x}\right|$$

LEAVE ON THE BOARD

Example 2:

$$\int \frac{x^{1/2}}{1+x^{3/4}}\,dx$$

One possibility would be to let

$x = u^4$ to eliminate the fractions in the exponents;

$dx = 4u^3$

So $\int \frac{x^{1/2}}{1+x^{3/4}}\,dx = \int \frac{u^2}{1+u^3}\,4u^3 du = \int \frac{4u^5}{1+u^3}\,du = \int \frac{4\left(u^2+u^5\right)-4u^2}{1+u^3}\,du =$

$$\int \frac{4\left(u^2+u^5\right)-4u^2}{1+u^3}\,du = \int 4u^2 - \frac{4u^2}{1+u^3}\,du = \frac{4u^3}{3} - \frac{4\ln\left|1+u^3\right|}{3}$$

But how to we evaluate

$$\int \frac{u^2}{1+u^3}\,du$$

We let $v = u^3$

Then $dv = 3u^2$

So $\int \frac{u^2}{1+u^3}\,du = \int \frac{1}{1+v}\frac{dv}{3} = \frac{1}{3}\ln\left|1+v\right| = \frac{1}{3}\ln\left|1+u^3\right|$

and therefore $\int 4u^2 - \frac{4u^2}{1+u^3}\,du = \frac{4u^3}{3} - \frac{4\ln\left|1+u^3\right|}{3}$

How does this work for Indefinite Integrals?

Let's look at example 1 again

First method:

$$\int_1^4 \frac{dx}{1+\sqrt{x}} = \left[2\sqrt{x} - \ln\left|1+\sqrt{x}\right|\right]_1^4 = 2\sqrt{4} - 2\ln\left|1+\sqrt{4}\right| - \left(2\sqrt{1} - 2\ln\left|1+\sqrt{1}\right|\right) =$$

$$4 - 2\ln 3 - \left(2 - 2\ln 2\right) = 2 + 2\ln\left(\frac{2}{3}\right) = 2 + \ln\frac{4}{9}$$

Or we can translate the limits:

$x = u^2$

if x=4, then u=2
if x=1, then u=1

$$\int_1^4 \frac{dx}{1+\sqrt{x}} = \left[2u - 2\ln\left|1+u\right|\right]_1^2 = 4 - 2\ln 3 - \left(2 - 2\ln 2\right) = 2 + 2\ln\frac{2}{3} = 2 + \ln\frac{4}{9}$$

$Letting\ u = g\left(x\right)$

$$\int_a^b f\left(g\left(x\right)\right)g'\left(x\right)dx = \int_{g(a)}^{g(b)} f\left(u\right)du$$

2nd Handout if Time