Lesson Plan 7 - Additional Techniques of Integration 5.7

1) Take attendance
2) Quiz Next Tuesday
3) Questions on Homework

Integration of odd powers of sine and cosine

$$\int \cos^3(x)\,dx = \int \cos^2(x)\cos(x)\,dx = \int \left(1-\sin^2(x)\right)\cos(x)\,dx =$$

$$\int \cos(x)\,dx - \int \sin^2(x)\cos(x)\,dx = \sin(x) - \frac{\sin^3(x)}{3} + C$$

$$\int \cos^5(x)\,dx = \int \cos^2(x)\cos^3(x)\,dx = \int \left(1-\sin^2(x)\right)\cos^3(x)\,dx =$$

$$\int \cos^3(x)\,dx - \int \sin^2(x)\cos^3(x)\,dx$$

$$\int \sin^2(x)\cos^3(x)\,dx = \int \sin^2(x)\cos^2(x)\cos(x)\,dx = \int \sin^2(x)\left(1-\sin^2(x)\right)(x)\cos(x)\,dx =$$

$$\int \sin^2(x)\cos(x) - \sin^4(x)\cos(x)\,dx = \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C$$

$$\int \cos^5(x)\,dx = \sin(x) - \frac{\sin^3(x)}{3} - \left[\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5}\right] + C =$$

$$\sin(x) - \frac{2\sin^3(x)}{3} + \frac{\sin^5(x)}{5} + C$$

For higher odd powers you would just repeat this.

For sine some signs change but otherwise the procedure is the same.

Even Powers of sine or cosine

Recall that $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$$\int \sin^2(x)\,dx = \frac{1}{2}\int 1 - \cos(2x)\,dx = \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right] + C = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

$$\int \sin^4(x)\,dx = \frac{1}{4}\int (1 - \cos(2x))^2\,dx = \frac{1}{4}\int 1 - 2\cos^2(2x) + \cos^4(2x)\,dx$$

For integrals such as $\int \sin^n(x)\cos^m(x)\,dx$,

if either *n* or *m* are odd, then using the Pythagorean identity it can be converted to terms like:

$$\sin^p(x)\cos(x)$$

or

$$\cos^q(x)\sin(x)$$

which are easily integrated. If both *n* and *m* are even, then it can be converted to terms which are all even powers of a sine or cosine, which we now know how to integrate.

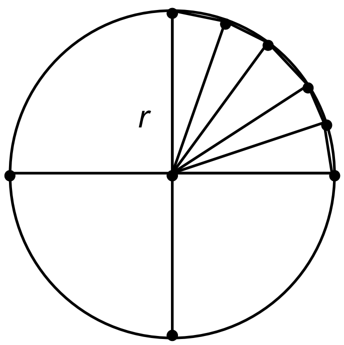
Similarly since $\sec^2(x) = 1 + \tan^2(x)$

$$\int \tan^3(x)\sec(x)\,dx = \int \tan(x)\left(\sec^2(x) + 1\right)\sec(x)\,dx =$$

$$\int \tan(x)\sec^3(x) + \int \tan(x)\sec(x)\,dx = \frac{\sec^3(x)}{3} + \sec(x) + C$$

Trigonometric Substitution:

Trying to integrate $\sqrt{a^2 - x^2}$ or $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ we can use a trigonometric substitution:



$x^2 + y^2 = r^2$

so

$y = \pm\sqrt{r^2 - x^2}$

$A = 4\int_0^r \sqrt{r^2 - x^2}$

Let $x = r\sin u$
$dx = r\cos u\, du$

$$\int \sqrt{r^2 - x^2}\,dx = \int \sqrt{r^2 - r^2\sin^2(u)}\,r\cos(u)\,du = \int r^2\cos^2(u)\,du$$

$$4\int_0^r \sqrt{r^2 - x^2}\,dx = 4r^2\int_0^{\pi/2}\cos^2(u)\,du = 4r^2\int_0^{\pi/2}\left(\frac{1+\cos(2u)}{2}\right)du = 4r^2\left[\frac{u+\sin(u)}{2}\right]_0^{\pi/2} =$$

$$4r^2\left[\frac{\pi/2+0}{2} + 0 - 0 - \sin(\pi)\right] = \pi r^2$$

Note for $\sqrt{a^2 + x^2}$ consider the substitution $x = a\tan(u)$ since
$a^2 + a^2\tan^2 = a^2\left(1+\tan^2\right) = \left(a\sec\right)^2$

Integration using Partial Fractions

We've done some simpler versions of these before, eg:

$$\int \frac{x}{1+x}\,dx = \int 1-\frac{1}{1+x}\,dx$$

What about something like

$$\int \frac{5x-4}{2x^2+x-1}\,dx \quad \text{Note that } \frac{d}{dx}2x^2+x-1 \neq A(5x-4)$$

Factoring the denominator

$$2x^2+x-1=(2x-1)(x+1)$$

So we set $\int \frac{5x-4}{2x^2+x-1}\,dx = \int \frac{A}{2x-1}+\frac{B}{x+1}\,dx$

To find A and B we work the following:

$$\frac{A}{2x-1}+\frac{B}{x+1}=\frac{A(x+1)+B(2x-1)}{(2x-1)(x+1)}=\frac{Ax+A+2Bx-B}{(2x-1)(x+1)}=\frac{x(A+2B)+(A-B)}{(2x-1)(x+1)}$$

Now we can see that $A+2B=5$ and $A-B=-4$
Subtracting the second from the first we have $3B=9$ or $B=3$ and therefore $A=-1$

$$\int \frac{5x-4}{2x^2+x-1}\,dx = \int \frac{-1}{2x-1}+\frac{3}{x+1}\,dx = -\frac{1}{2}\ln|2x-1|+3\ln|x+1| = \ln\left|\frac{(x+1)^3}{\sqrt{2x-1}}\right|+C$$

Note that if a term in the denominator is raised to a power, you need an extra term, eg.

$$\frac{x}{(x+2)^2(x-1)} = \frac{A}{(x+2)^2} + \frac{B}{(x+2)} + \frac{C}{x-1}$$

If a factor is irreducible, eg.

$$\frac{1}{(1+x^2)(x-1)} = \frac{Ax+B}{1+x^2} + \frac{C}{x-1} \quad A = B = -\frac{1}{2}, \quad C = \frac{1}{2}$$

$$\int \frac{1}{(1+x^2)(x-1)}\,dx = \frac{1}{2}\int \frac{-(x+1)}{(1+x^2)} + \frac{1}{x-1}\,dx = \frac{1}{2}\int \frac{-x}{1+x^2} + \frac{-1}{1+x^2} + \frac{1}{x-1}\,dx =$$