Lesson Plan 8 - Approximate Integration 5.9

1) Take attendance
2) Quiz Next Tuesday, Midterm May 19th
3) Questions on Homework
4) Review Previous Worksheet
Need Video for PC and Overhead

We've already looked at doing approximate integration using
left endpoint and right endpoint integration.
Today we will learn about a few more accurate methods.

But first, why do we need to learn about Approximate integration.
The book suggests two reasons.

First, some integrals do not have an anti-derivative.

The classic example is $\int e^{-x^2}dx$.
There are many others, but this integral is important in statistics and there very important for business and the social sciences.

Whoever today it is very easy to evaluate this as a definite integral .
You might want to note that $\left[\int_{-\infty}^{\infty} e^{-x^2}dx\right]^2 = \pi$.

Show this on a graphic calculator.

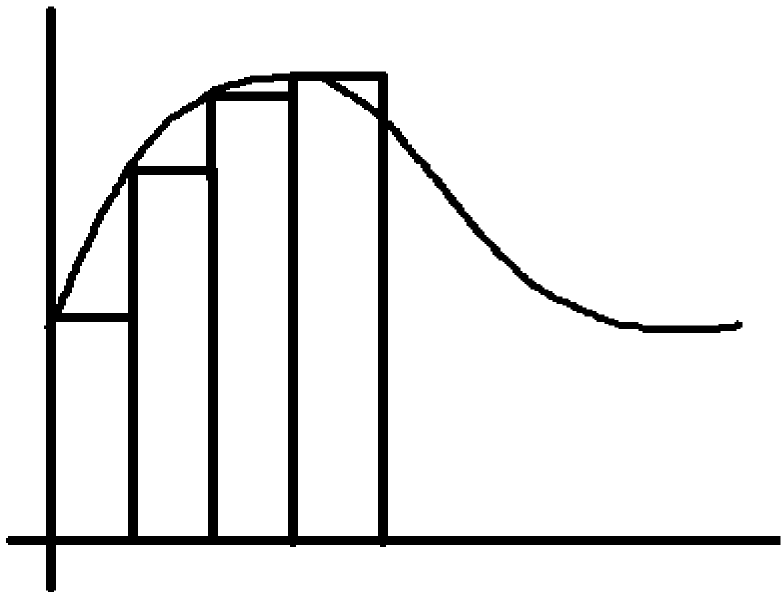
So why would we need to do an approximate integration?

If you are computer programmer and you need to write some code like the code your graphics calculator uses, you will need to know more about how it is doen.

Also, if you are collecting data points for a function, and you want to integrate that function, then you will need one of the methods we will describe.
This is described in Example 5 p 408 in the book where someone has recorded the data rate over a period of time, and wants to know how much data was transmitted.

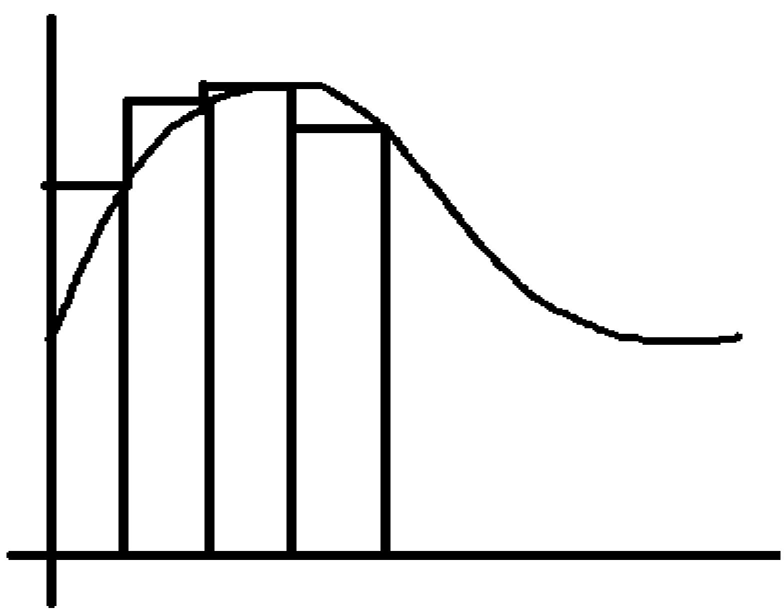
In this case, you don't have a mathematical description of the function, just some data points.

Left end point:



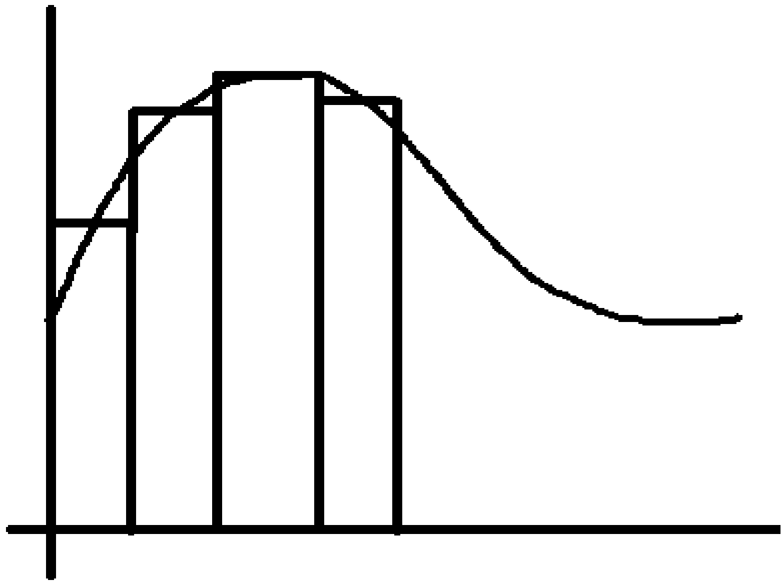
$$A = \Delta x \sum_{i=0}^{n-1} f\left(x_{i+1}\right) \text{where } \Delta x = \frac{x_n - x_0}{n-1}$$

Right end point:



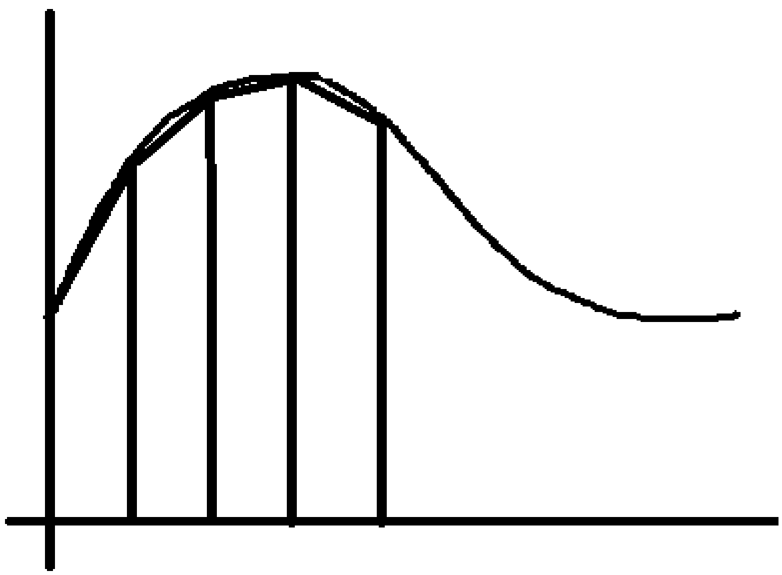
$$A = \Delta x \sum_{i=1}^{n} f\left(x_i\right) \text{ where } \Delta x = \frac{x_n - x_0}{n-1}$$

Midpoint Rule:



$$A = \Delta x \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \text{ where } \Delta x = \frac{x_n - x_0}{n-1}$$

Trapezoidal Rule:



Area of a trapezoid is $A = \frac{w_1 + w_2}{2} h$ so we get

$$A = \Delta x \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \text{ where } \Delta x = \frac{x_n - x_0}{n-1}$$

Let's compare these methods for a couple of functions:

$\int_1^2 \frac{1}{x} dx$ letting n=5

Note the exact value is $A_{exact} = \ln(2) - \ln(1) = .693147$

$x_i = \{1.0, 1.25, 1.5, 1.75, 2.0\}$ so

$f(x_i) = \{1.0, 0.8, 0.66666667, 0.57142857, 0.5\}$

Using the website calculator we find that

$A_{left} = .75952381$

$A_{right} = .63452381$

$A_{trapezoid} = .69702381$

For the midpoint rule we can't use the website because it doesn't know the function:

$$A = \left(\frac{1}{4}\right)\left[\frac{1}{1.125} + \frac{1}{1.375} + \frac{1}{1.625} + \frac{1}{1.875}\right] = .6912198912$$

Left	Right	MidPoint	Trapezoid	Simpson	Exact
.75952381	.63452381	.6912198912	.69702381		.6931471806
.066	.059	.0019	.0039		

What can we conclude looking at these answers?

Both Midpoint and Trapezoid are better than left and right, but Midpoint is better than Trapezoid.

Error Bounds

The book provides a way of estimating a maximum error for Midpoint and Trapezoid.

Suppose $\left|f''(x)\right| \le K \ \ for\ a \le x \le b$

Then the errors E_T and E_M can be bounded by

$$\left|E_T\right| \le \frac{K(b-a)^3}{12n^2} \text{ and } \left|E_M\right| \le \frac{K(b-a)^3}{24n^2}$$

This indicates that the maximum error when using the Trapezoidal rule is twice that when using the Midpoint rule.

Let's examine this with respect to our calculation.

$$\left|f''(x)\right| = \frac{2}{x^3} \le 2 \text{ so}$$

$$.0039 = \left|E_T\right| \le \frac{2(2-1)^3}{12(4)^2} = .0104 \qquad .0019 = \left|E_M\right| \le \frac{2(2-1)^3}{24(4)^2} = .0052$$

Note that the actual errors are still smaller than the maximum.

If we wanted the Midpoint error to be less than .00001 what does *n* have to be?

We want $\frac{2}{24n^n} < .00001$ so $n > \sqrt{\frac{2}{24 \cdot .00001}} = 91.3$ so we would need *n*=92 to guarantee this accuracy.

Why would is this calculation useful?

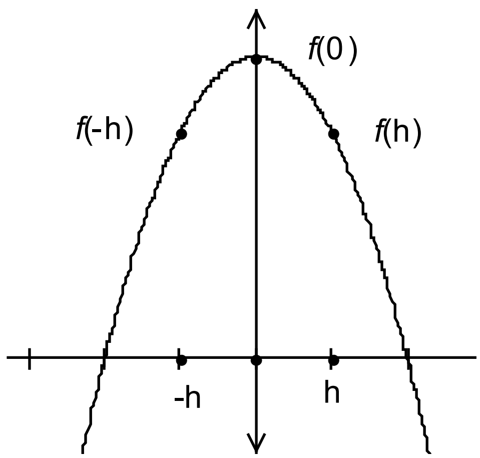
If you want to write a computer code that runs on a calculator, you want it to run as fast as possible. So you want to know at what point your calculation would exceed the accuracy of numbers used.

Pass out worksheet

A better approximation called Simpson's rule approximates the function between each group of three points using a parabola.   The resulting sum is surprisingly simple:

$$A_{parabola} = \frac{f(x_0) + 4f(x_1) + f(x_2)}{3} \Delta x$$

This is suspiciously simple and we should verify it.



Consider the parabolic equation and the area underneath it

$$Area = \int_{-h}^{h} Ax^2 + Bx + C\ dx = \int_{-h}^{h} Ax^2 + C\ dx + \int_{-h}^{h} Bx\ dx$$

Note that the first term is an even function and the 2nd term is an odd function. So we may rewrite this as

$$Area = 2\int_{0}^{h} Ax^2 + C\ dx + 0 = 2\left[\frac{Ax^3}{3} + Cx\right]_0^h = 2\left(\frac{Ah^3}{3} + Ch\right) = \frac{h}{3}\left(2Ah^2 + 6C\right)$$

Now note what happens when we add

$$f(-h) + 4f(0) + f(h) = Ah^2 - Bh + C + 4C + Ah^2 + Bh + C = 2Ah^2 + 6C$$

So $Area = \frac{h}{3}\left(f(-h) + 4f(0) + f(h)\right)$

We can translate this parabola left and right to any three points, so we can sum as follows:

$$Area = \frac{\Delta x}{3} \sum_{i=0}^{i=} \left(f(x_{i*3}) + 4f(x_{i*3+1}) + f(x_{i*3+2})\right)$$

Let's now try the website calculator again:

Left	Right	MidPoint	Trapezoid	Simpson	Exact
.75952381	.63452381	.6912198912	.69702381	.693253778	.6931471806
.066	.059	.0019	.0039	.00001	

Note the error for Simpson's rule is given by:

$$.000001 = \left|E_S\right| \le \frac{K\left(b-a\right)^5}{180n^4} = .000017$$

So Simpson's rule give a much more accurate approximation.

Finish problems on worksheet