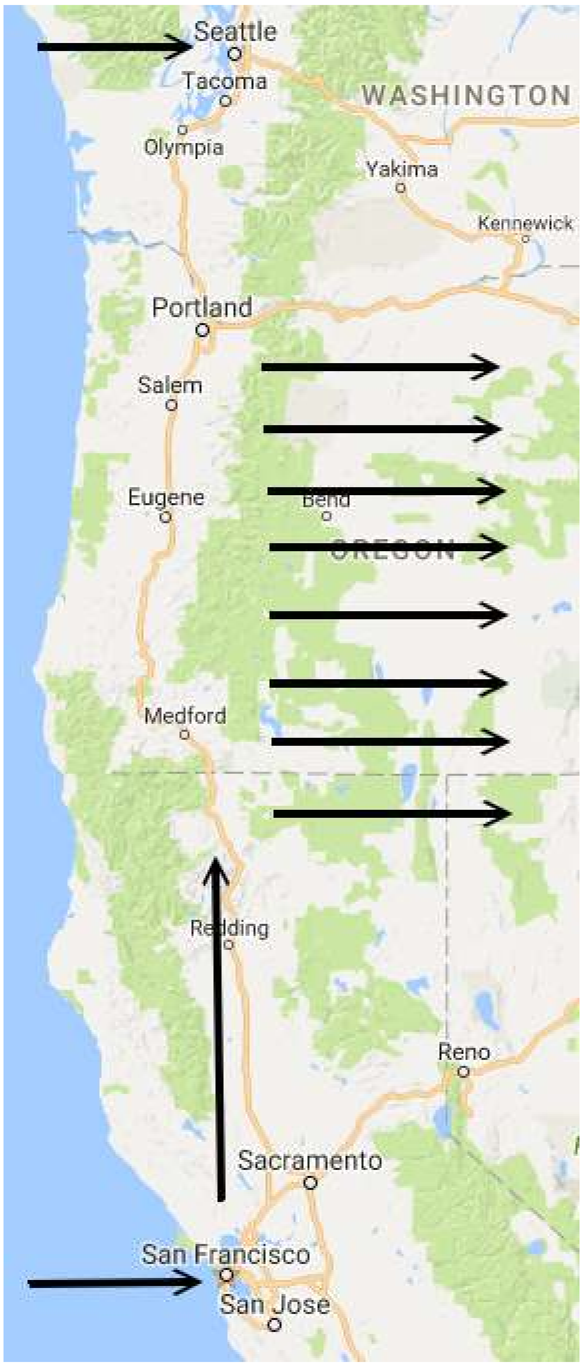
Lesson Plan 13 Vectors Math 48C Mitchell Schoenbrun

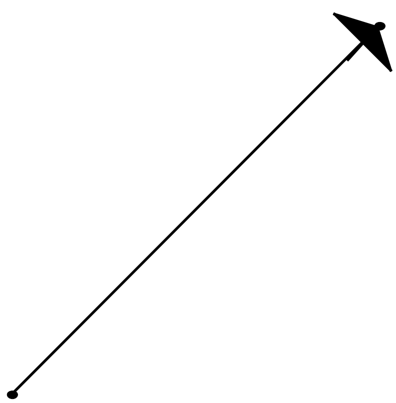
If you were to fly an airplane from San Francisco to Seattle at 200mph, you might try heading directly North. If along the way you encountered a constant Easterly wind going 30 mph, you would end up far East of your destination. Clearly you want to head somewhat North and West, but in which directly exactly? To answer this question we are going to investigate **vectors**.



A vector is a mathematical object, somewhat different from anything we've seen previously, although it has many properties similar to numbers.

Like a number, a vector is an abstract object that has no physical reality, and also like numbers it is very useful for modeling the physical world.

We represent a vector graphically as an arrow.



A vector has a starting point called the **tail** and an ending point called the **head** or **tip**.

We will often label a vector with a letter and write it rotationally with an arrow above it.

$$\vec{V}$$

If the vector is describe by two points A the tail and B the head, we will write it this way:

$$\overrightarrow{AB}$$

Note that the order of A and B is important.

A vector has two numerical values associated with it

1) A magnitude or size which is always positive. When we draw a vector graphically, the length of the vector represents the magnitude. A technical name for the magnitude is the

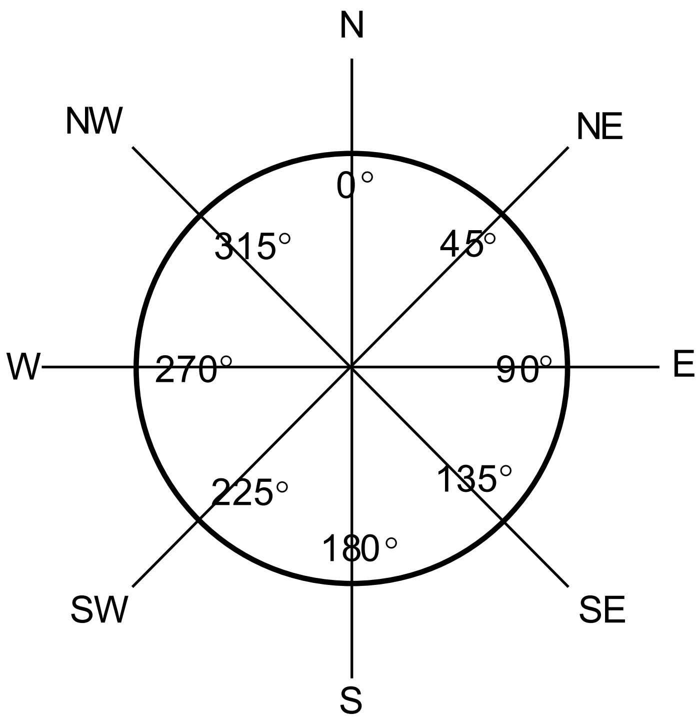
NORM of a vector, which we write this way:

$$\left|\vec{V}\right| \qquad \text{OR} \qquad \left\|\vec{V}\right\|$$

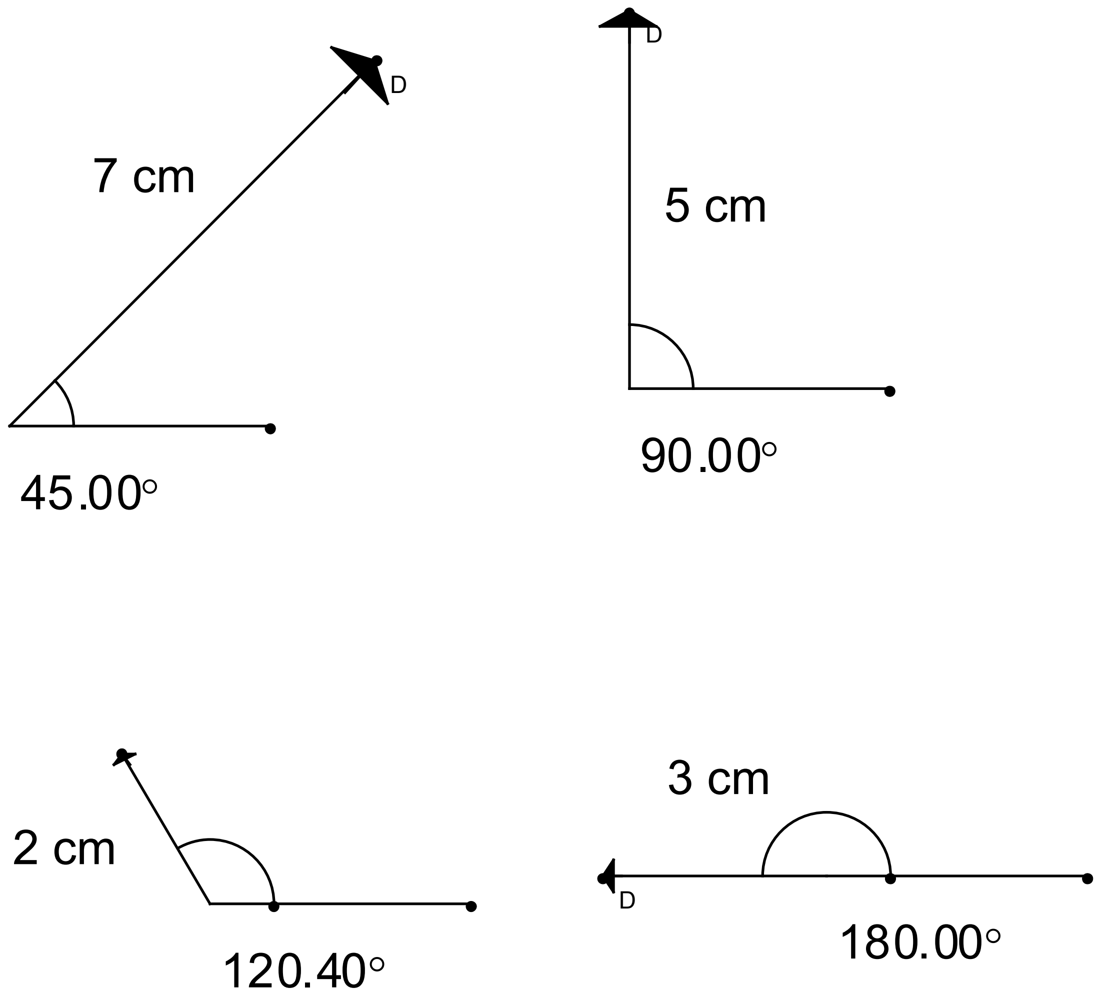
2) it also has direction. The direction is represented graphically by the angle of the vector relative to the x-axis.

A note about using vectors in aviation:

Vectors are used a lot in aviation and some of our examples and problems will involve airplanes, so it is important to know that the angle is represented a little differently. 0° represents North and is straight up. Angles increase in value from 0° to 360° clockwise:



SOME GRAPHICAL EXAMPLES:

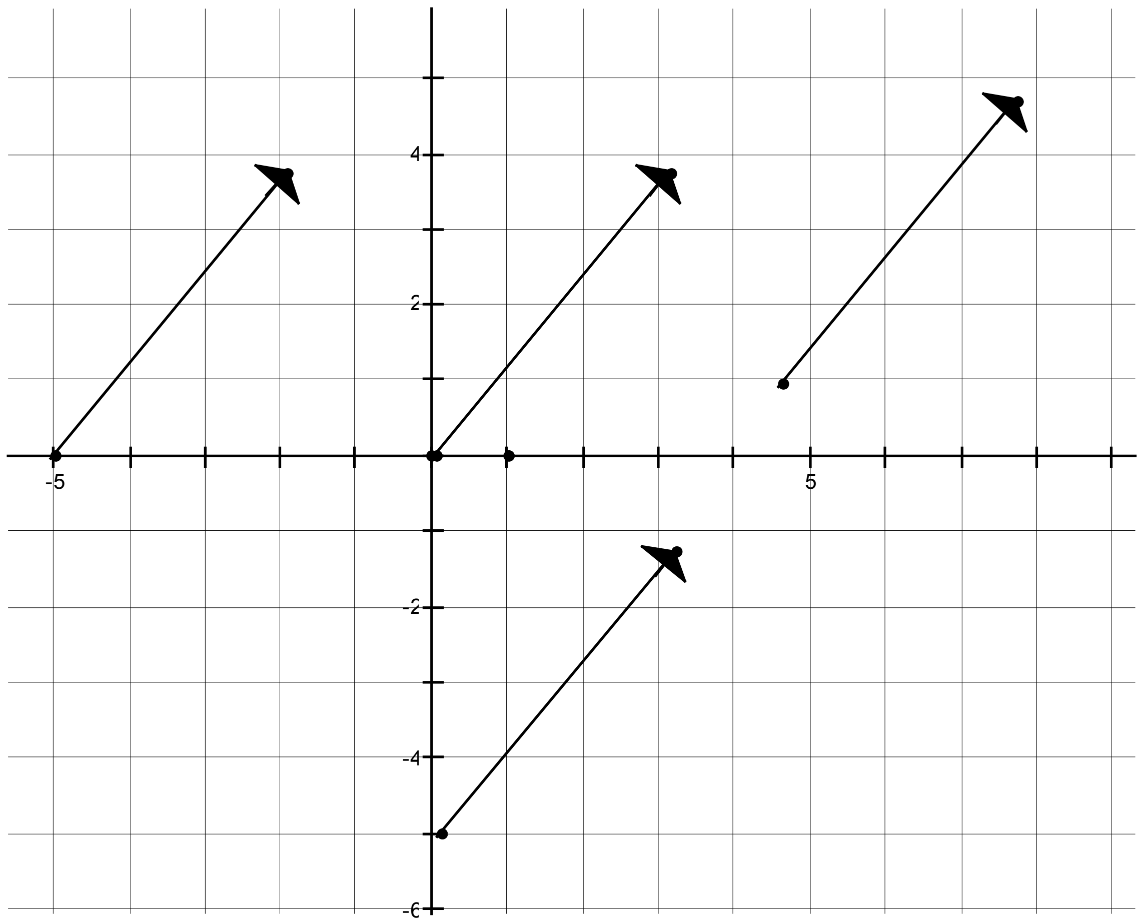


A vector does not have an absolute position the way a line segment would.

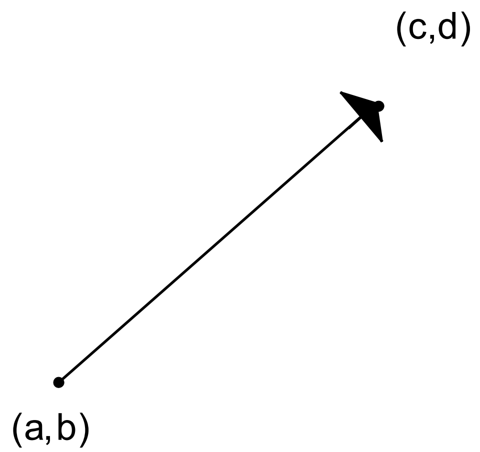
When we represent a vector graphically on a Cartesian grid the particular location we choose is just representative of the vector.

All such arrows that we might draw are equivalent representatives of the vector.

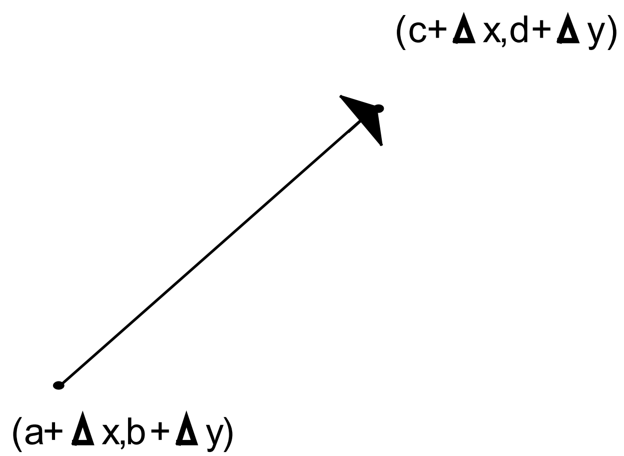
That is we can move the arrow around on the grid without changing the vector.



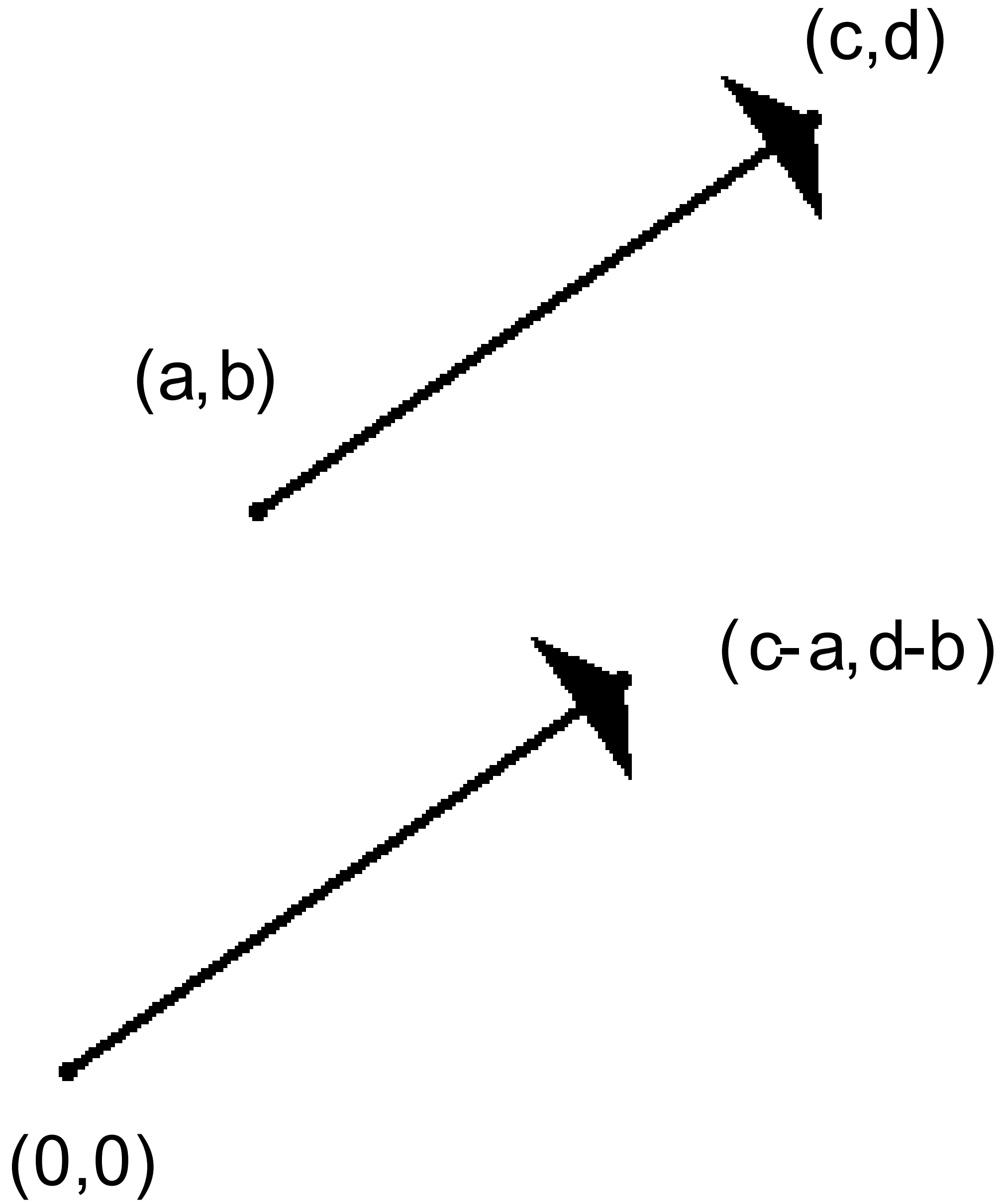
When we draw a vector on a Cartesian grid, the tail and head will have coordinates:



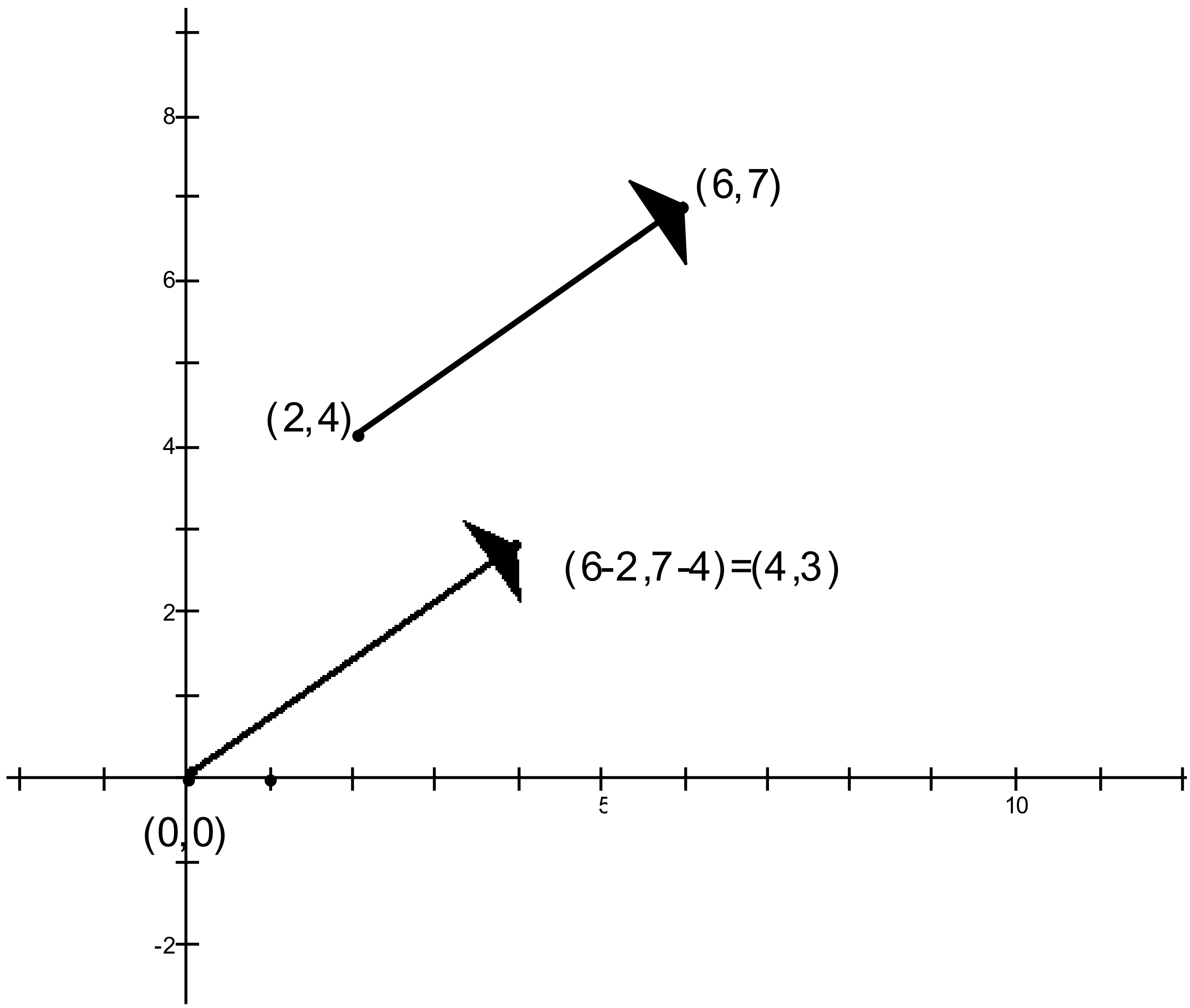
We can move the vector around by adding the same offset to the x and/or y component.



Give the coordinates of a vector's head and tail, we will often move a vector so that it's tail is at the origin.   This is easy to do as follow:



AN EXAMPLE OF MOVING A VECTOR'S TAIL TO THE ORIGIN



NOTE THAT THIS DOES NOT CHANGE THE MAGNITUDE OR DIRECTION.

A vector with its tail at the origin is called a **position vector**.

If we know a vector is a position vector:

$$\overrightarrow{V} = (0,0) \rightarrow (5,7)$$

The Origin is redundant so we can simply indicate the vector this way using angle brackets.

$$\overrightarrow{V} = \langle 5,7 \rangle$$

This is called **Vector Component Form.**

The component form of a vector has unique components.

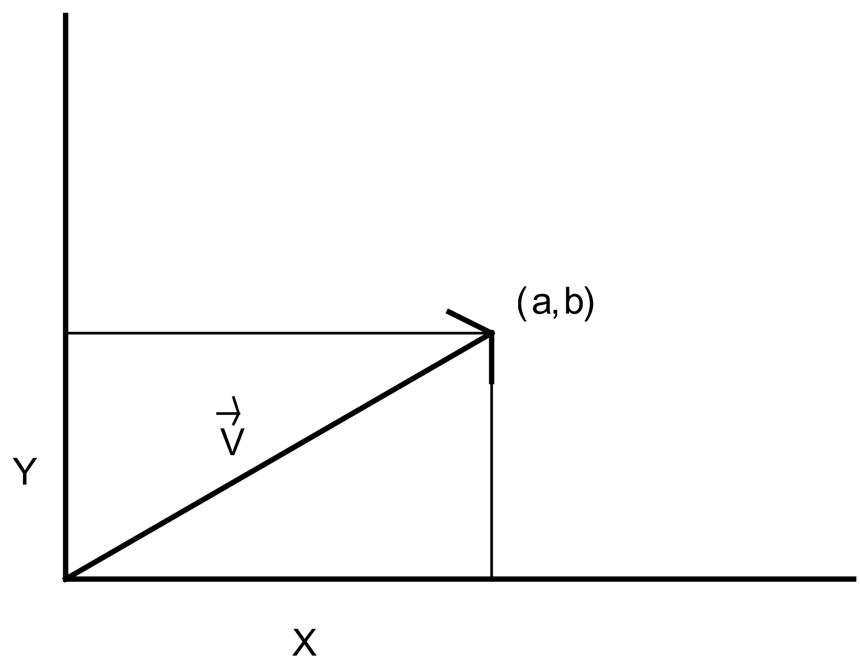
Calculating the Magnitude and Angle of a vector from it's components?

Looking at this diagram, it is easy to see that the Magnitude of the of vector $\vec{V}$ can be computed from its components using the Pythagorean theorem.

$$\left|\vec{V}\right| = \sqrt{a^2 + b^2}$$

Also it's direction can be found using our knowledge of trigonometry

$$\angle\vec{V} = \arctan\left(\frac{y}{x}\right)$$



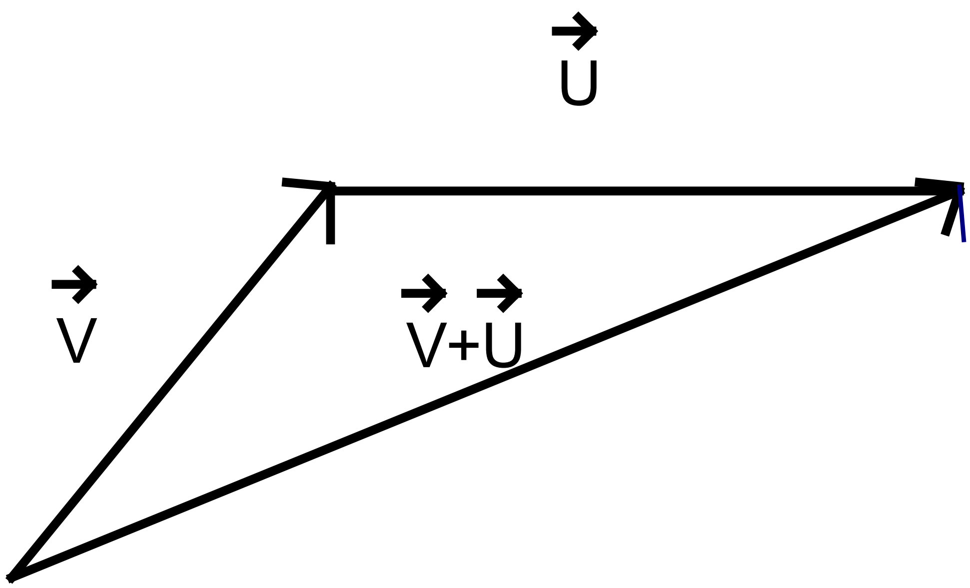
Note that as before you much check the sign(s) of x and y to decide which quadrant the angle is in.

ADDING VECTORS

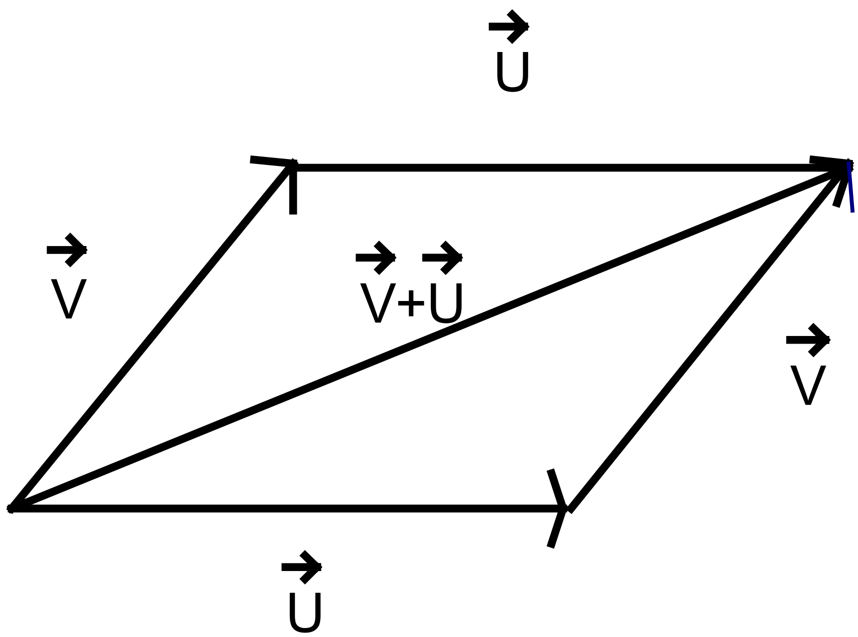
Like simple numbers we can add any two vectors and get a new vector.
The new vector is called **THE RESULTANT**.

There are two equivalent ways to do this.

Graphically - If we want to add two vectors $\vec{V}$ and $\vec{U}$ we move $\vec{U}$ so that its tail is coincident with the head of $\vec{V}$. The new vector $\vec{V}+\vec{U}$ has the tail of $\vec{V}$ and the head of $\vec{U}$.



This is sometimes called the parallelogram method. This diagram shows why?



Note that Vector Addition is Commutative, that is:

$$\overrightarrow{U}+\overrightarrow{V}=\overrightarrow{V}+\overrightarrow{U}$$

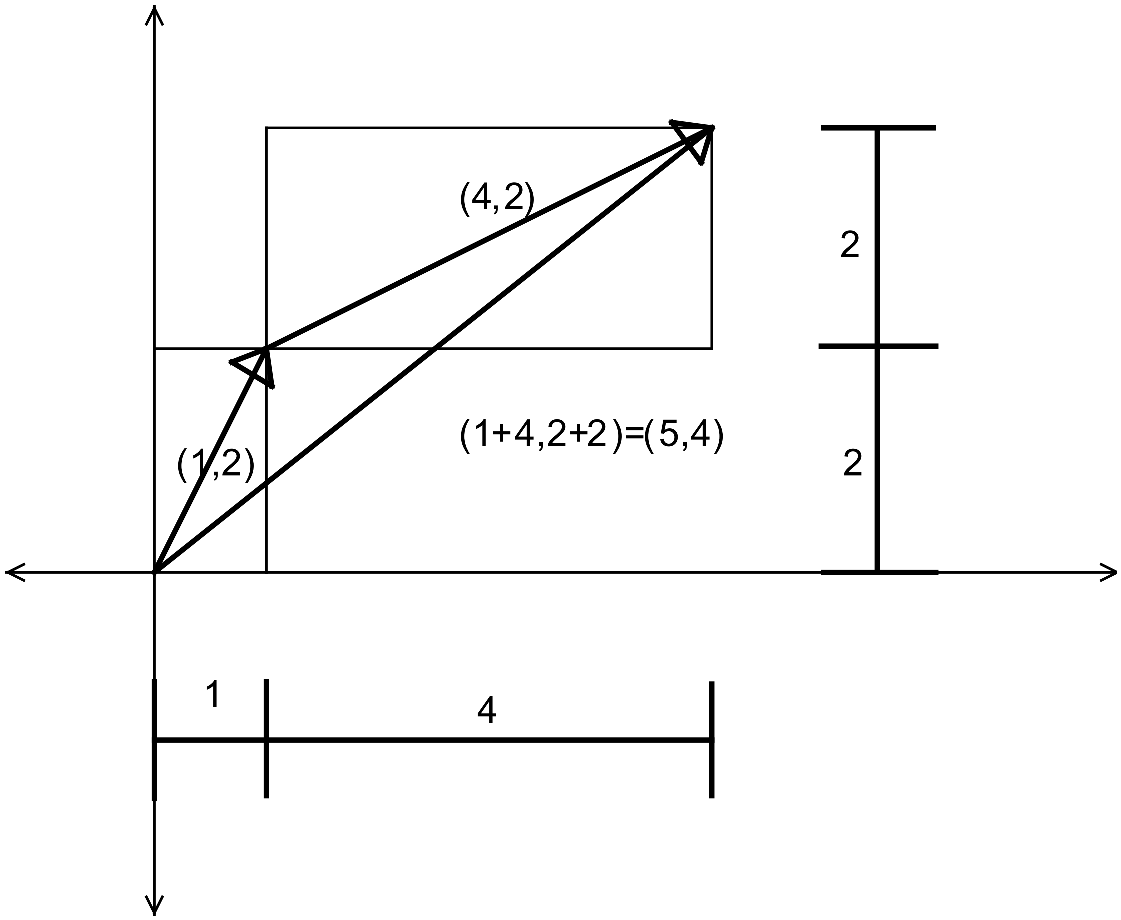
The second way to add vectors is algebraically by adding their components.

If $\vec{V} = \langle a, b \rangle$ and $\vec{U} = \langle c, d \rangle$

then

$$\vec{V} + \vec{U} = \langle a + c, b + d \rangle$$

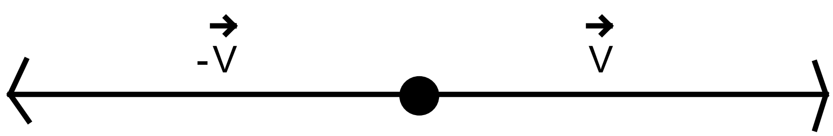
It's easy to see that these are equivalent using an example on a graph:



We subtract two vectors by adding the opposite of the subtracted vector.

$$\vec{V} - \vec{U} = \vec{V} + -\vec{U}$$

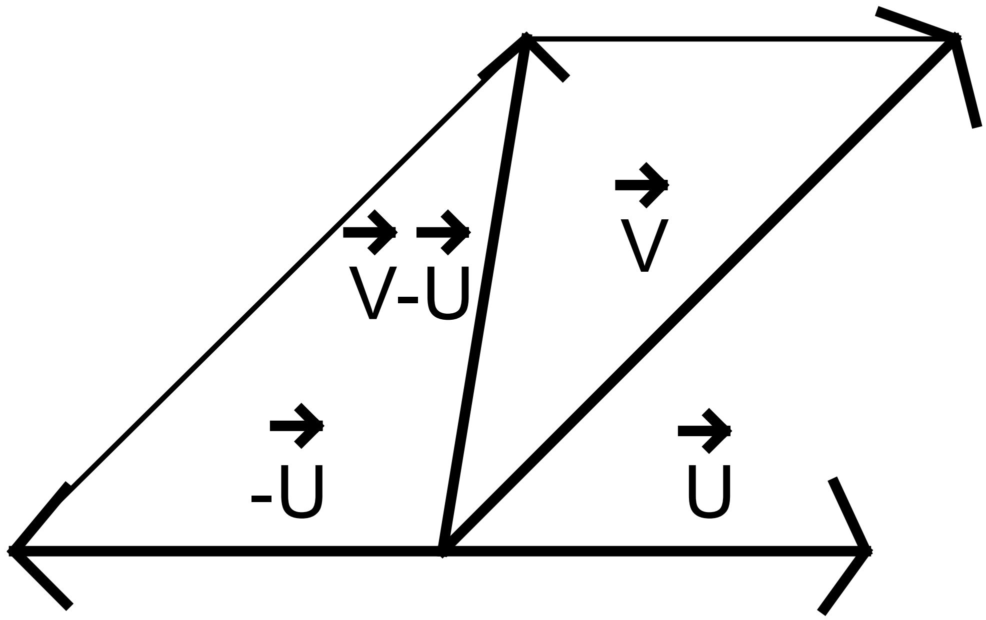
The opposite of a vector is a vector with the same magnitude in the opposite direction.



Note that if $\vec{V} = \langle a, b \rangle$

then $-\vec{V} = \langle -a, -b \rangle$

Subtracting Two Vectors looks like this graphically



Algebraic similarities between addition of Vectors and Numbers.

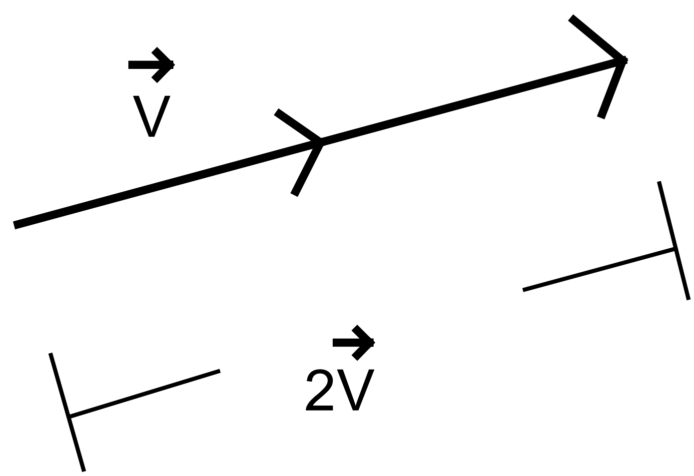
Numbers	Vectors	Property
Any two numbers can be added	Any two vectors can be added	Closure property
(a+b)+c = a+(b+c)	$\left(\vec{V}+\vec{U}\right)+\vec{W}=\vec{U}+\left(\vec{V}+\vec{W}\right)$	Associative property
a+b = b+a	$\vec{V}+\vec{U}=\vec{U}+\vec{V}$	Commutative property
There is a number Zero such that a + 0 = a	There is a vector called the zero vector and written $\vec{0}$ such that $\vec{V}+\vec{0}=\vec{V}$	Identity property
Every number a has an inverse -a such that a+ -a = 0	Every vector $\vec{V}$ has an inverse $-\vec{V}$ such that $\vec{V}+-\vec{V}=\vec{0}$	The Inverse property

Vectors have some additional properties.

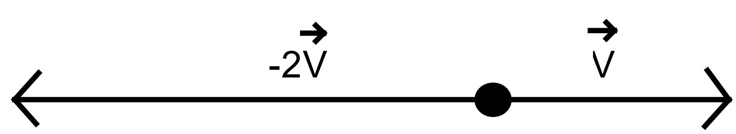
Vectors can be multiplied by a number.   This is called **SCALAR** multiplication.

A scalar is just another name for a real number.

Here is how we multiply a vector graphically.



Or



Notice that if you multiply a vector by a negative number, the new vector is in the opposite direction.

Multiplying vectors in component form is even easier:

$$k \cdot \langle a, b \rangle = \langle ka, kb \rangle$$

Note that multiplying a vector by -1 is the same as finding it's opposite or inverse, which is the same for numbers.

The distributive laws of Scalar multiplication over vector addition.

$$k \cdot \left(\overrightarrow{V} + \overrightarrow{U} \right) = k\overrightarrow{V} + k\overrightarrow{U}$$

This is easy to prove using component form.

$$k \cdot \left(\overrightarrow{V} + \overrightarrow{U} \right) =$$
$$k \cdot \left(\left\langle V_x, V_y \right\rangle + \left\langle U_x, U_y \right\rangle \right) =$$
$$k \cdot \left\langle V_x + U_x, V_y + U_y \right\rangle =$$
$$\left\langle k \cdot \left(V_x + U_x \right), k \cdot \left(V_y + U_y \right) \right\rangle =$$
$$\left\langle kV_x + kU_x, kV_y + kU_y \right\rangle =$$
$$\left\langle kV_x, kV_y \right\rangle + \left\langle kU_x, kU_y \right\rangle$$
$$k \cdot \left\langle V_x, V_y \right\rangle + k \cdot \left\langle U_x, U_y \right\rangle$$
$$k\overrightarrow{V} + k\overrightarrow{U}$$

A second distributive law looks like this.

$$\left(k + l \right) \cdot \overrightarrow{V} = k\overrightarrow{V} + l\overrightarrow{V}$$

A Unit Vector is any vector whose magnitude is 1

$$\left|\overrightarrow{V_{unit}}\right| = 1$$

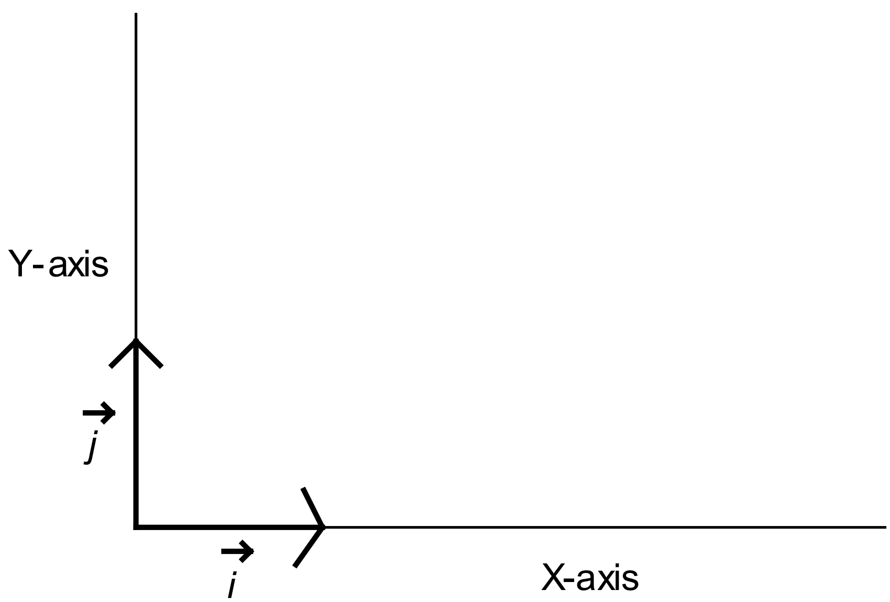
It's good to know that is easy to change any vector into a unit vector with the original vector's direction by multiplying it by 1 over the norm:

$$\overrightarrow{V_{unit}} = \frac{1}{\left|\vec{V}\right|} \cdot \vec{V}$$

There are two special unit vectors that we using both mathematics and science a lot:

$$\vec{i} = \langle 1, 0 \rangle$$

$$\vec{j} = \langle 0, 1 \rangle$$



What is handy about $\vec{i}$ and $\vec{j}$ is that any vector in component form can be written as a linear combination of $\vec{i}$ and $\vec{j}$ as follows:

$$\vec{V} = \langle a, b \rangle = a\vec{i} + b\vec{j}$$

A) Do Example 2 Vectors

B) Do Example 5 Force

Prove second distributive Law

Go Over in class

Homework 1-4, 11-21, 36, 39, 44, 45