Lesson Plan 14 Vectors, the Dot Product 48C Mitchell Schoenbrun

1) Attendance
2) homework questions

We've studied some basic properties of vectors.

We've found that like numbers vectors can be added, subtracted and multiplied by another number.

The question arises as to whether there is any way to combine two vectors in a product.

There are two ways to do this called the Dot Product and the Cross Product.

These are written as follows: (see next page).

We will not be covering the cross product in this class, but anyone taking Physics will come across it, so we will take a quick look at the cross product near the end of class if there is time.

$$\vec{V} \cdot \vec{U}$$

Dot-Product

$$\vec{V} \times \vec{U}$$

Cross-Product

We start our discussion of the Dot Product with the idea of a projection:

This is section 9.2 on Page 589 in our book,
a different 9.2!

We project one vector onto another.

Here Vector V is projected onto Vector U (next page)

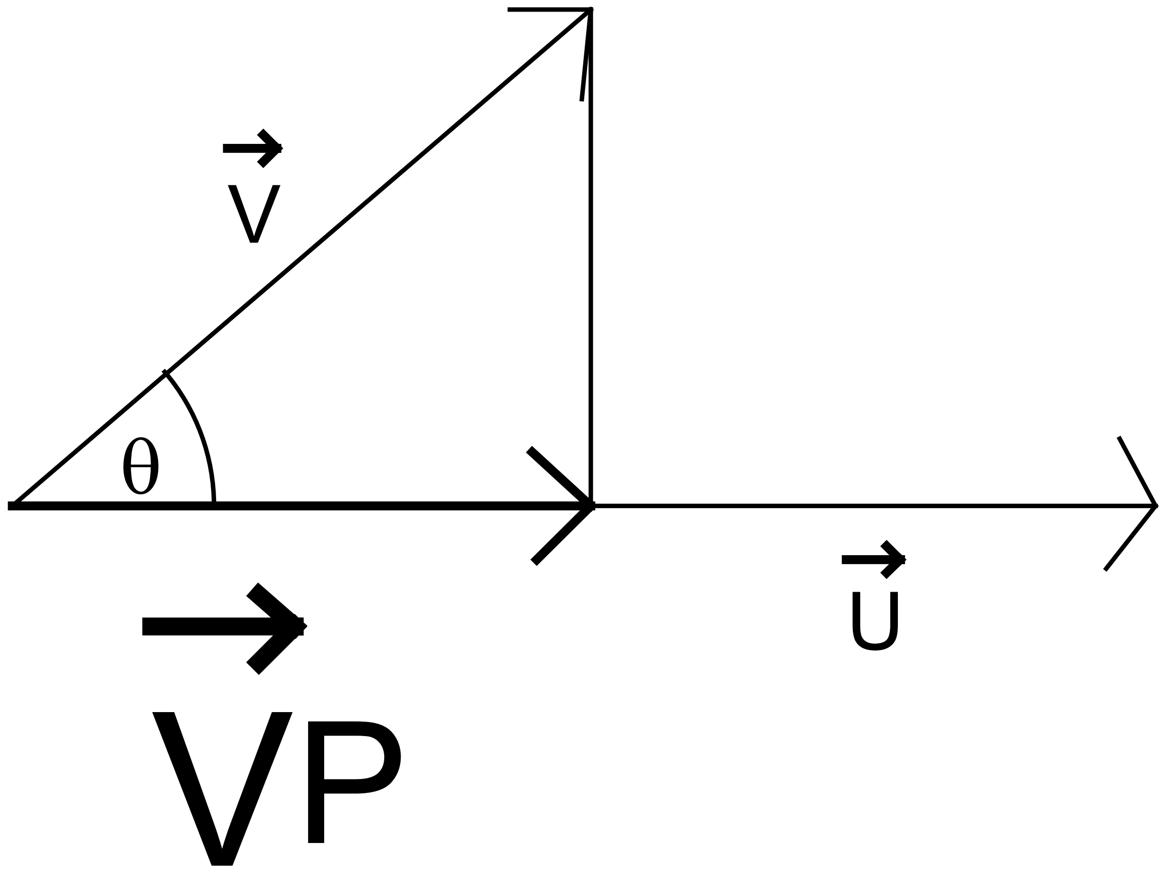
Note that:

$$\left\| \overrightarrow{V_P} \right\| = \left\| \vec{V} \right\| \cos(\theta)$$

Also $\overrightarrow{V_P}$ has the same direction as $\vec{U}$

Recall from Wednesday that we can make $\vec{U}$

into a Unit Vector, a vector with magnitude 1 as follows:


$\vec{V}$
θ
$\vec{U}$
$\vec{V}P$

$$\overrightarrow{U}_{Unit} = \frac{\overrightarrow{U}}{\left\|\overrightarrow{U}\right\|}$$

This gives us a definition of a Projection:

$$\overrightarrow{V}_{P \to \overrightarrow{U}} =$$

$$\left\|\overrightarrow{V}\right\| \cos(\theta) \frac{\overrightarrow{U}}{\left\|\overrightarrow{U}\right\|} = \frac{\left\|\overrightarrow{V}\right\|}{\left\|\overrightarrow{U}\right\|} \cos(\theta) \overrightarrow{U}$$

We can now define the Dot Product of

$\vec{V}$ and $\vec{U}$ as

$$\vec{V} \cdot \vec{U} = \left\|\vec{U}\right\| \left\|\vec{V}_{P \to \vec{U}}\right\| =$$

$$\left\|\vec{V}\right\| \left\|\vec{U}\right\| \cos(\theta)$$

Also note that projection can be re-defined as:

$$\vec{V}_{P \to \vec{U}} = \left(\frac{\vec{V} \cdot \vec{U}}{\left\|\vec{U}\right\|^2} \right) \vec{U}$$

Important Properties of the Dot Product

The Dot Product of two Vectors is a Number, not a Vector.

The Dot Product is Commutative.

$$\vec{V} \cdot \vec{U} = \vec{U} \cdot \vec{V}$$

If two vectors are parallel, then $\cos(\theta) = 1$ and

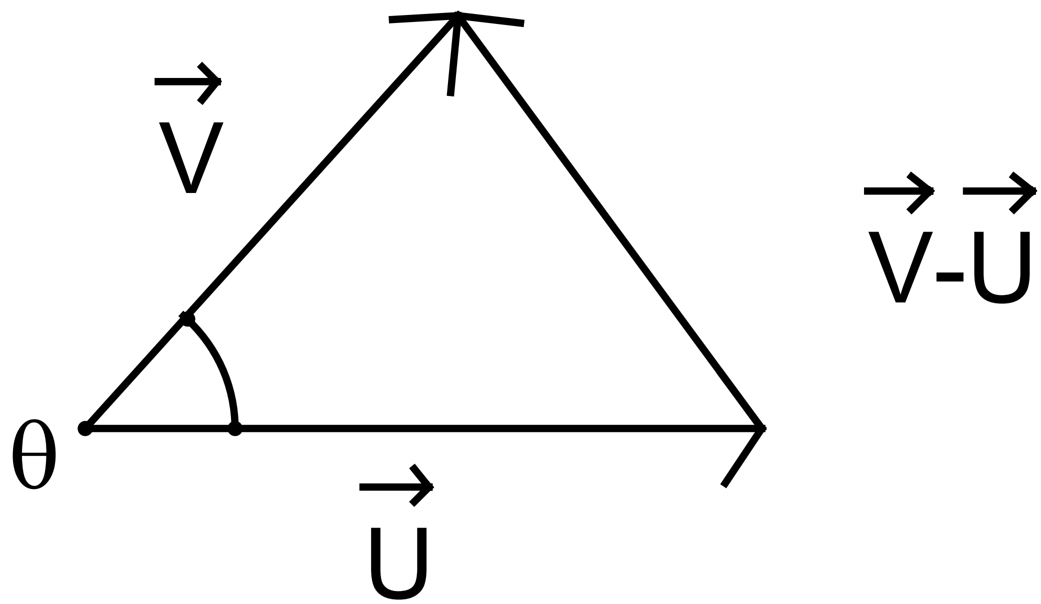
$$\vec{V} \cdot \vec{U} = \left\|\vec{V}\right\| \left\|\vec{U}\right\|$$

If two vectors are perpendicular $\cos(\theta) = 0$

$$\vec{V} \cdot \vec{U} = 0$$

If we knew the Dot Product of two vectors, we could immediately calculate the measure of the angle between them.

Recall this diagram for subtracting vectors



Applying the Law of Cosines to the norms of the sides and simplifying we get the following proof:

$$C^2 = A^2 + B^2 - 2AB\cos(\theta)$$

$$\left\|\vec{V} - \vec{U}\right\|^2 = \left\|\vec{V}\right\|^2 + \left\|\vec{U}\right\|^2 - 2\left\|\vec{V}\right\|\left\|\vec{U}\right\|\cos(\theta)$$

If we define the components of *V* and *U* as

$\vec{V} = \langle V_x, V_y \rangle \quad \vec{U} = \langle U_x, U_y \rangle$ we get the

following proof:

$$\left[\sqrt{\left(V_x - U_x\right)^2 + \left(V_y - U_y\right)^2}\right]^2 =$$

$$\left[\sqrt{V_x{}^2 + V_y{}^2}\right]^2 + \left[\sqrt{U_x{}^2 + U_y{}^2}\right]^2 - 2\left\|\vec{V}\right\|\left\|\vec{U}\right\|\cos\left(\theta\right)$$

$$\left(V_x - U_x\right)^2 + \left(V_y - U_y\right)^2 = V_x{}^2 + V_y{}^2 + U_x{}^2 + U_y{}^2 - 2\left\|\vec{V}\right\|\left\|\vec{U}\right\|\cos\left(\theta\right)$$

$$V_x^2 - 2V_xU_x + U_x^2 + V_y^2 - 2V_yU_y + U_y^2 =$$

$$V_x{}^2 + V_y{}^2 + U_x{}^2 + U_y{}^2 - 2\left\|\vec{V}\right\|\left\|\vec{U}\right\|\cos\left(\theta\right)$$

$$-2\left(V_xU_x + V_yU_y\right) = -2\left\|\vec{V}\right\|\left\|\vec{U}\right\|\cos\left(\theta\right)$$

$$\left\|\vec{V}\right\|\left\|\vec{U}\right\|\cos\left(\theta\right) = V_xU_x + V_yU_y$$

This Gives us an easy way to calculate the dot product from the vector components.

$$\overrightarrow{V} \cdot \overrightarrow{U} = V_x U_x + V_y U_y$$

Note that the pattern here persists into 3 dimensions

$$\overrightarrow{V} \cdot \overrightarrow{U} = V_x U_x + V_y U_y + V_z U_z$$

What can we do with this?

Since

$$\overrightarrow{V} \cdot \overrightarrow{U} = V_x U_x + V_y U_y = \left\| \overrightarrow{V} \right\| \left\| \overrightarrow{U} \right\| \cos(\theta)$$

We can calculate θ as follows:

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{V}\cdot\overrightarrow{U}}{\left\|\overrightarrow{V}\right\|\left\|\overrightarrow{U}\right\|}\right) =$$

$$\cos^{-1}\left(\frac{V_x U_x + V_y U_y}{\sqrt{V_x^2 + V_y^2}\sqrt{U_x^2 + U_y^2}}\right)$$

Examples from the book:

$$\vec{u} = \langle 3, 5 \rangle$$

$$and$$

$$\vec{v} = \langle 2, -8 \rangle$$

What is the angle between the two vectors?

$$\theta = \cos^{-1}\left(\frac{3\cdot 2 + 5\cdot -8}{\sqrt{34}\cdot\sqrt{68}}\right) = \cos^{-1}\left(\frac{-34}{\sqrt{34\cdot 68}}\right) = 135^\circ$$

$$\vec{u} = \langle 2, 1 \rangle$$

$$and$$

$$\vec{v} = \langle -1, 2 \rangle$$

What is the angle between the two vectors?

Since $2 \cdot -1 + 1 \cdot 2 = 0$ $\cos^{-1}(0) = 90^{\circ}$

So the vectors are perpendicular!

Example from the book

Find the components of *u* along *v*

$$\vec{u} = \langle 1, 4 \rangle$$

$$and$$

$$\vec{v} = \langle -2, 1 \rangle$$

What are the components of the projection of *u* onto *v*?

$$\vec{u}_{P\to\vec{v}} = \left(\frac{\vec{u}\cdot\vec{v}}{\left\|\vec{u}\right\|^{2}}\right)\vec{v} =$$

$$\frac{1\cdot -2+4\cdot 1}{\left(\sqrt{4+1}\right)^{2}}\left\langle -2,1\right\rangle =$$

$$\frac{2}{5}\left\langle -2,1\right\rangle = \left\langle \frac{-4}{5},\frac{2}{5}\right\rangle$$

We will talk more about finding components on Wednesday.

Do Worksheet and go over:

The norm of the cross product of two vectors is as follows:

$$\left\|\overrightarrow{V} \times \overrightarrow{U}\right\| = \left\|\overrightarrow{V}\right\| \left\|\overrightarrow{U}\right\| \sin(\theta)$$