Lesson Plan 15 Vectors, Work and the Cross Product 48C Mitchell Schoenbrun

1) Attendance
2) homework questions

For this section, I will have to introduce some concepts from Physics that you may not be familiar with. I will try to minimize the impact of this.

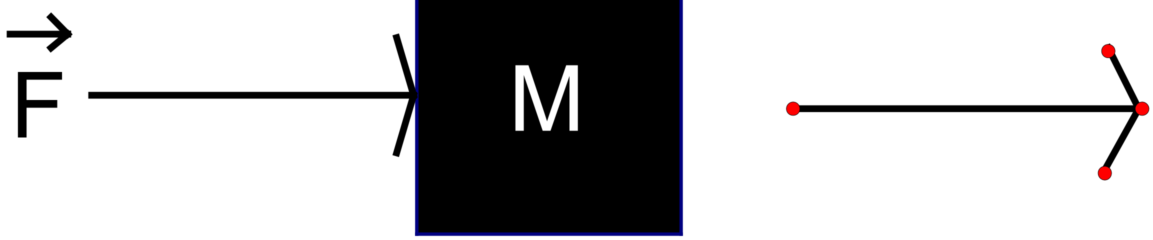
First a few concepts from physics.

A force is a vector quantity that has the units Mass * Distance/$Time^2$

An important way to view this is Mass * Acceleration.

Physics views a force as a quantity that when applied to a mass will accelerate it in the direction of the force vector.

$$\vec{A} = \frac{\vec{F}}{M}$$



When a force is applied to a mass and it accelerates over a distance, Mechanical energy is transferred to the mass. The transferred energy is know technically as Work.

Both Energy and Work have the same units:
Mass * $Distance^2/Time^2$

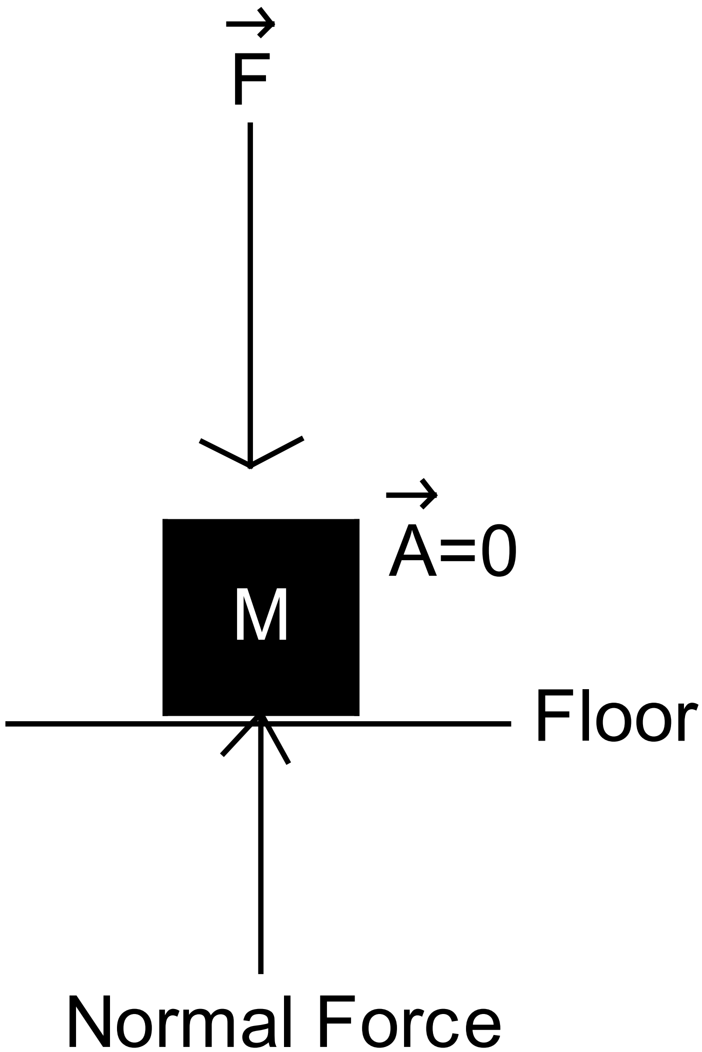
There are two different ways to look at these units:

1) Force * Distance = Mass * Acceleration * Distance
This is what we call **WORK**

2) Mass * $Velocity^2$
This is what we call **MECHANICAL ENERGY**

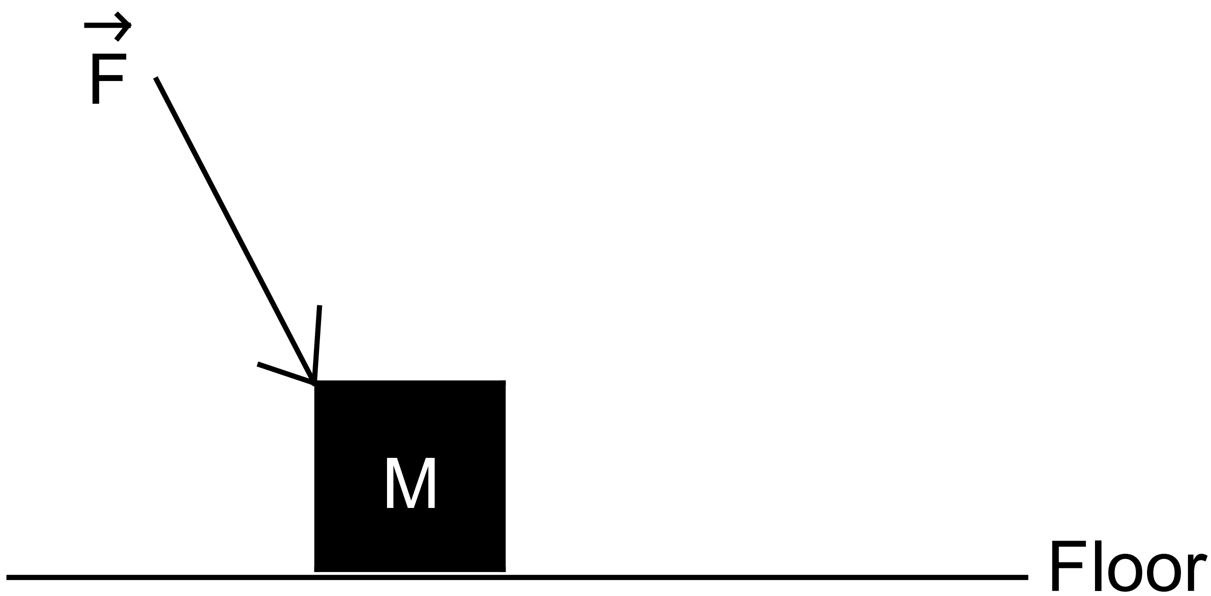
One counter-intuitive feature of this interaction that surprises many first time physics students is that if a force is applied to something that doesn't move, no work is done, and no energy is transfered.

One way to think about this is that if the mass doesn't move, there must be another force balancing the first force exactly. The net force is zero. The balancing force is often called a **NORMAL** force.

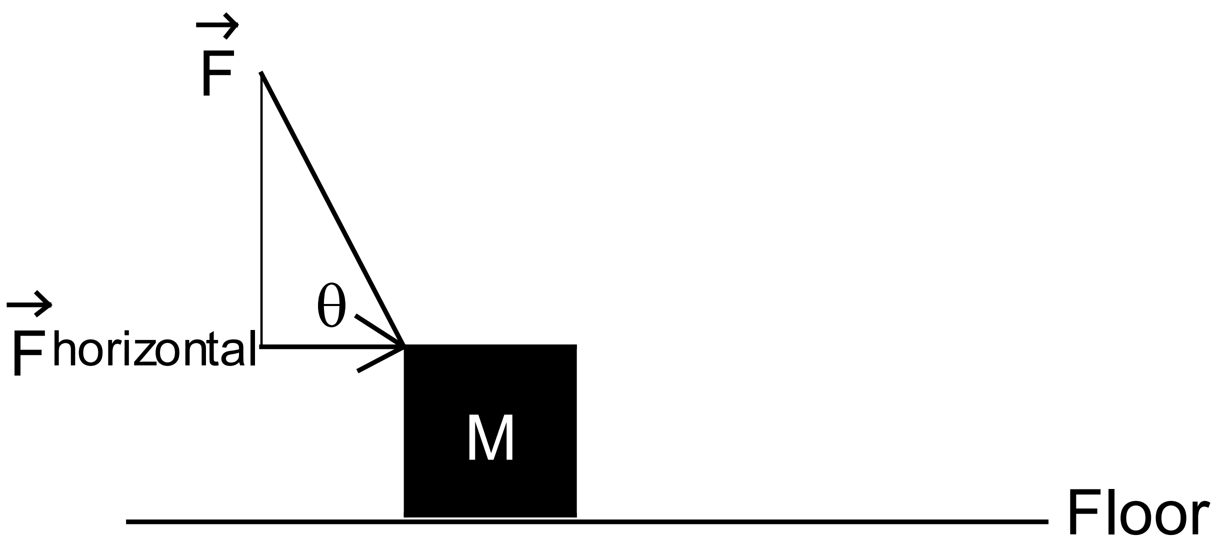


$\vec{F}$
$\vec{A}=0$
M
Floor
Normal Force

The question we are going to look at, is what happens when a force is applied to an object at an angle to the direction that the object can move.



What we find is that component of the force in the direction that the mass can move is the force that will accelerate the mass.   The rest of the force is balanced by a normal force.



$$\vec{F}_{horizontal} = \vec{F}\cos(\theta)$$

Writing the Work equation gives us the following:

$$Work = \left\|\overrightarrow{F}\right\|\left\|\overrightarrow{D}\right\|\cos\left(\theta\right) = \overrightarrow{F} \cdot \overrightarrow{D}$$

So our Dot product comes in handy here.

$$Work = \overrightarrow{F} \cdot \overrightarrow{D}$$

This is the first case we've found where we can use the dot product on vectors signifying different physical quantities.

Recall that we can calculate our Dot product easily if we know the components of the Force and Distance vector:

$$Work = F_x D_x + F_y D_y$$

Example 7: from the book

A force is given by the vector $\vec{F} = \langle 2,3 \rangle$
and moves an object from point $(1,3)$ to the point $(5,9)$

The displacement vector is $(5-1, 9-3) = (4,6)$

So the work done is

$$W = \vec{F} \cdot \vec{D} = (2,3) \cdot (4,6) = 8 + 18 = 26$$

Example 8 from the book

A man pulls a wagon exerting a force of 20lb on the handle at an angle of 60° for a distance of 100 feet. What is the work done?

$$W = \left\|\overrightarrow{F}\right\| \left\|\overrightarrow{D}\right\| \cos\theta = 20 \cdot 100 \cos\left(60^{\circ}\right) = 1000\, ft - lbs.$$

Worksheet Problems

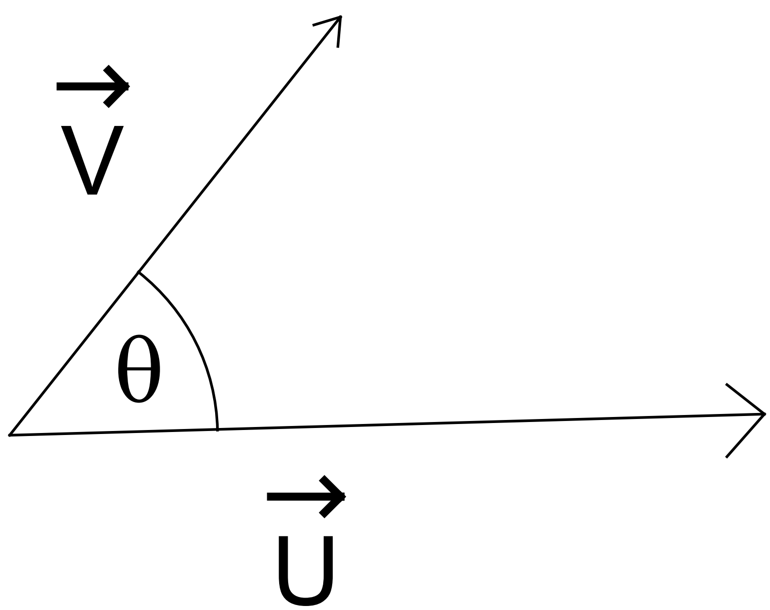
$\overrightarrow{V} \cdot \overrightarrow{U}$ Dot-Product

$\overrightarrow{V} \times \overrightarrow{U}$ Cross-Product

The norm of the cross product of two vectors is as follows:

$$\left\|\vec{V}\times\vec{U}\right\| = \left\|\vec{V}\right\|\left\|\vec{U}\right\|\sin(\theta)$$

where θ is the angle between the two vectors:



This means that when the two vectors are parallel,

$$\theta = 0^\circ$$

$$or$$

$$\theta = 180^\circ$$

so $\sin(\theta) = 0$

so the length of the cross product is Zero

If the two vectors are perpendicular

$$\theta = 90^\circ$$

$$or$$

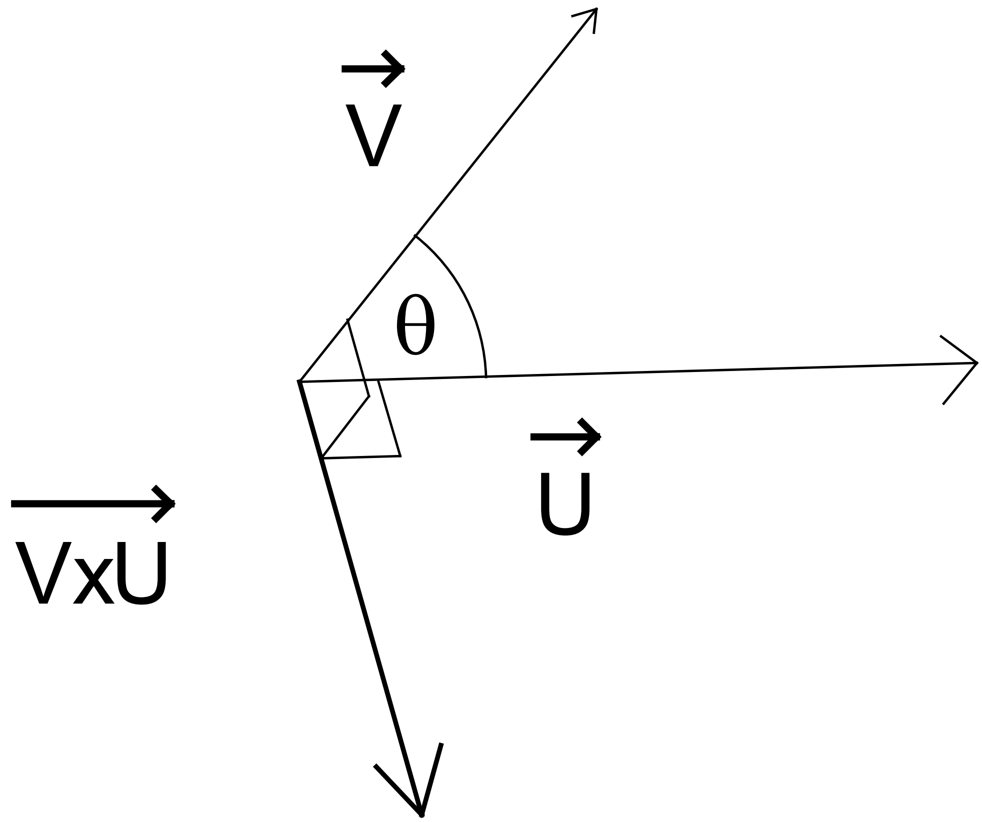
$$\theta = 270^\circ$$

so $\sin(\theta) = 1$

and

$$\left\|\vec{V} \times \vec{U}\right\| = \left\|\vec{V}\right\|\left\|\vec{U}\right\|$$

The direction of the cross product is always perpendicular to both vectors:



$$\vec{V} \times \vec{U} \perp \vec{V}$$

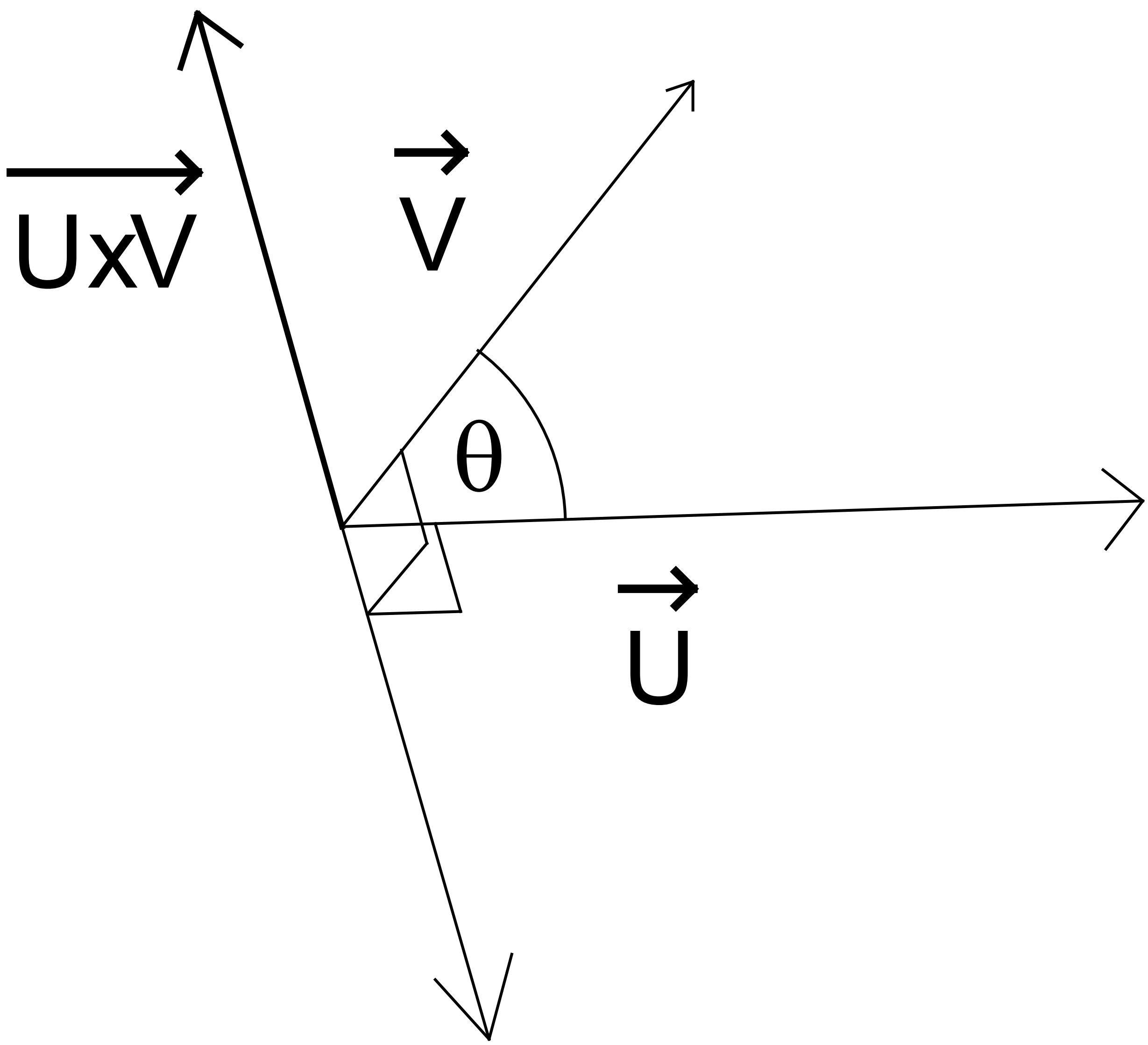
and

$$\vec{V} \times \vec{U} \perp \vec{U}$$

The Cross Product is not commutative.

$$\overrightarrow{V \times U} \neq \overrightarrow{U \times V}$$

When reversing the order, the direction of the cross product vector reverses.



$\overrightarrow{U \times V}$
$\vec{V}$
θ
$\vec{U}$

So what purpose does this strange product provide?

Here is an example from Physics,
Precession of the Gyroscope:

What is happening