Lesson Plan 18 Trigonometric Identities III, Math 48C Mitchell Schoenbrun

1) Attendance

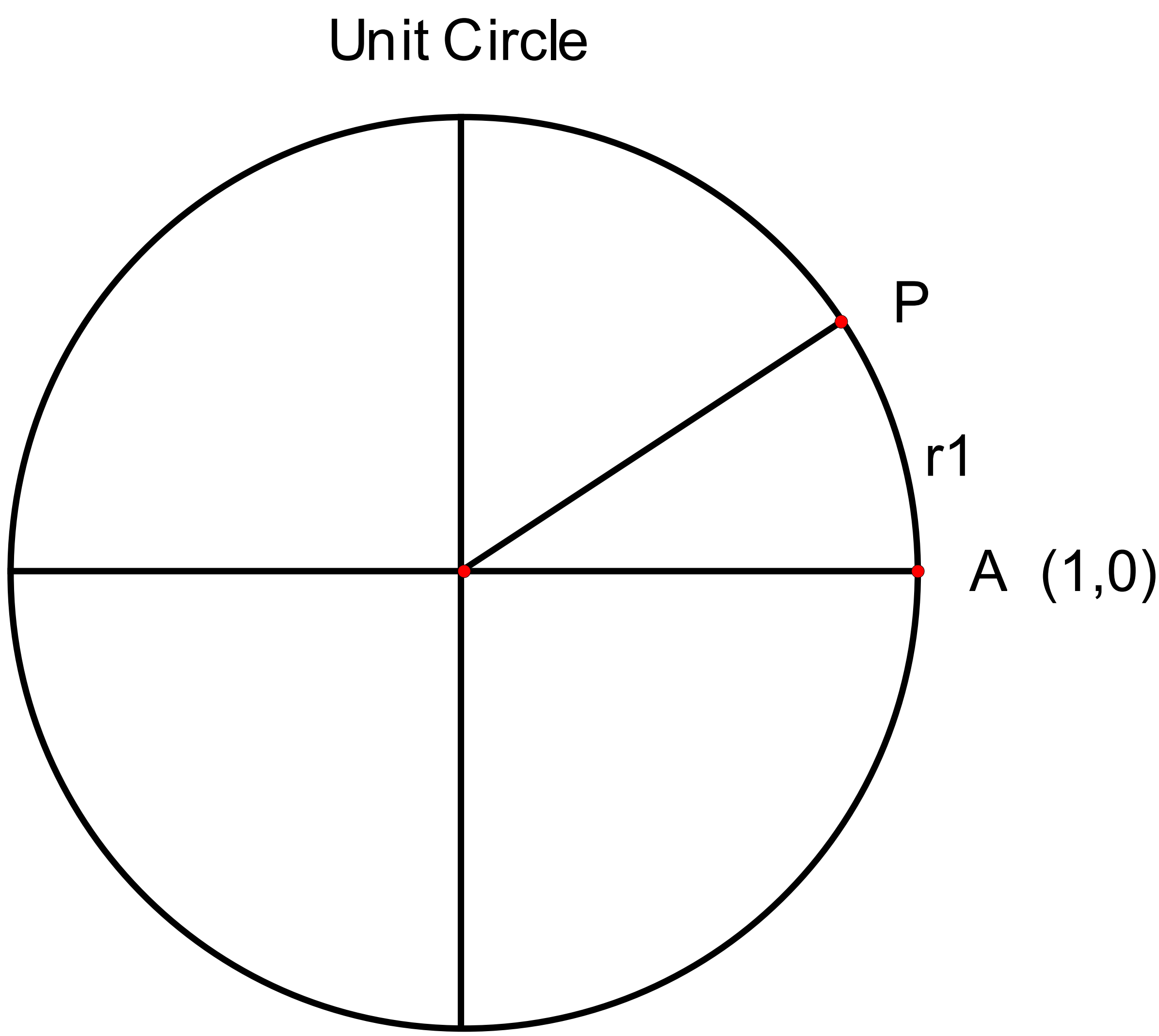
We're going to search for a summation formulae for the cosine of the sum of two angles:

$$\cos(x+y)=?$$

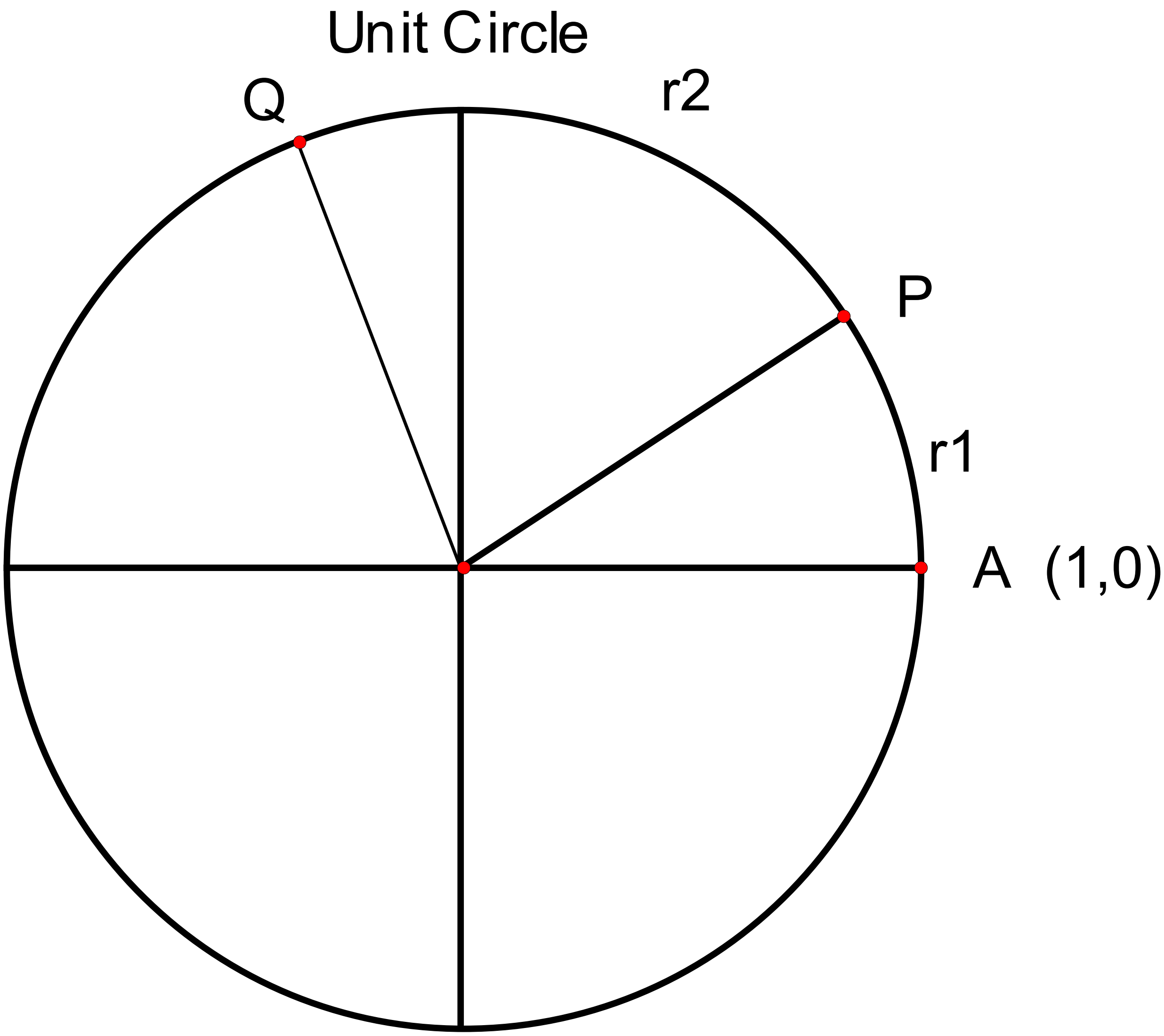
The proof we will show is a little obscure. You are not responsible for the steps, however you will need to know the result.

We start with a unit circle with point A at the coordinates (1,0) and point P an arbitrary point in the first quadrant.

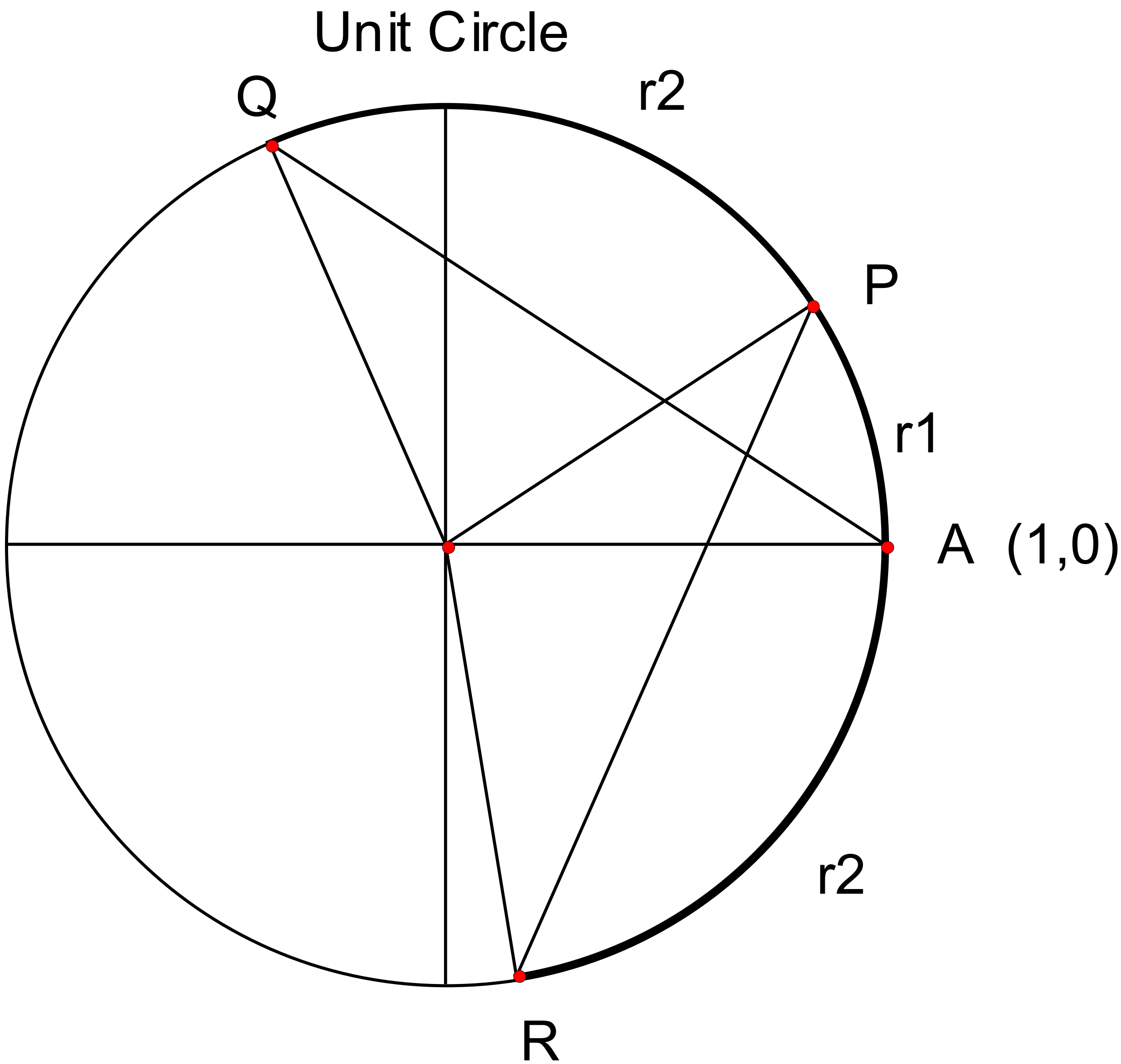
We label the angular size of arc $\widehat{AP}$ r1



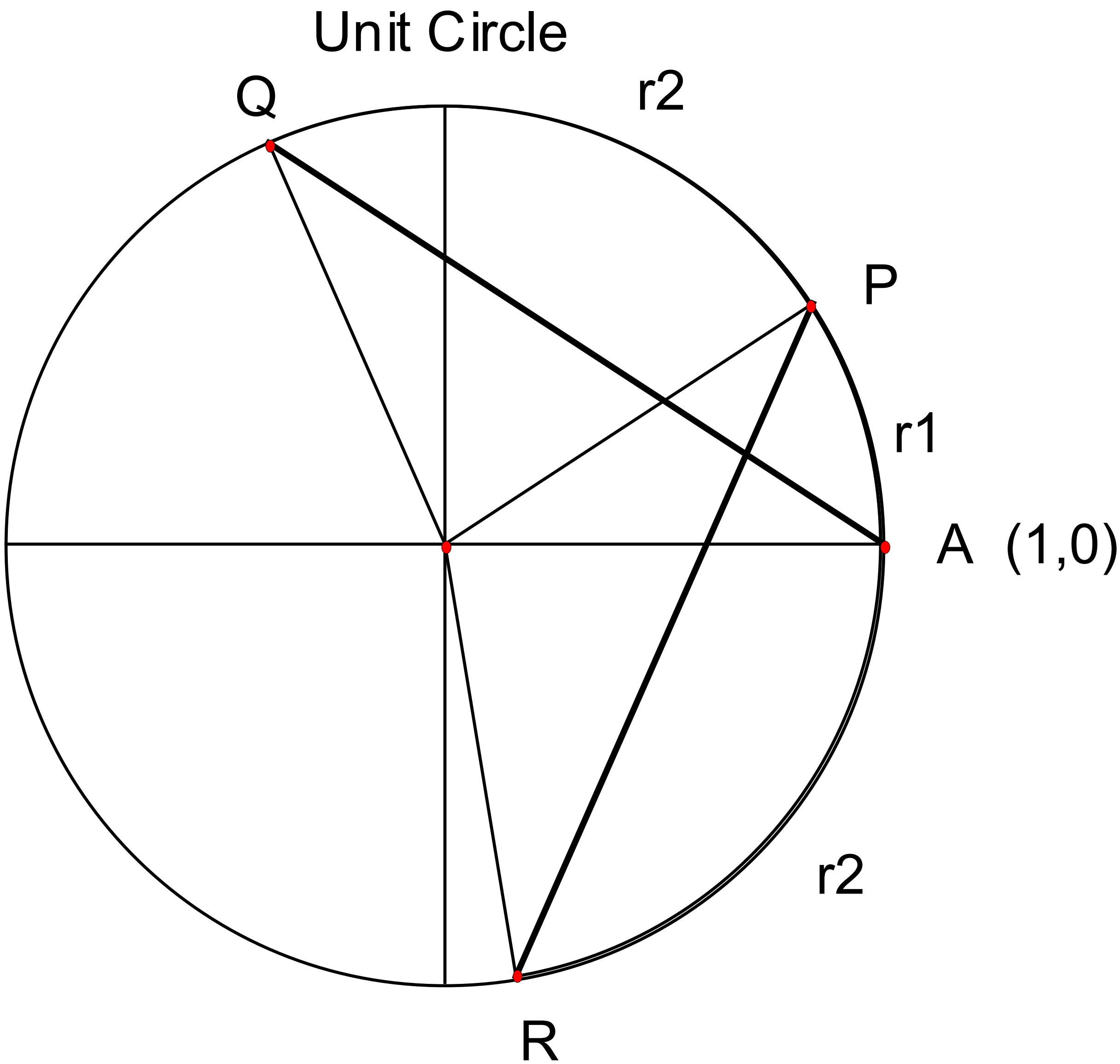
Next we add an arbitrary point Q in the 2nd quadrant and label the angular size of arc $\overset{\frown}{PR}$ r2.



Next we construct a point R in the fourth quadrant such that $\overset{\frown}{PQ} = \overset{\frown}{AR}$



Since r1+r2 = r2+r1, we have $\widehat{AQ} = \widehat{PR}$ and so also $\overline{AQ} = \overline{PR}$



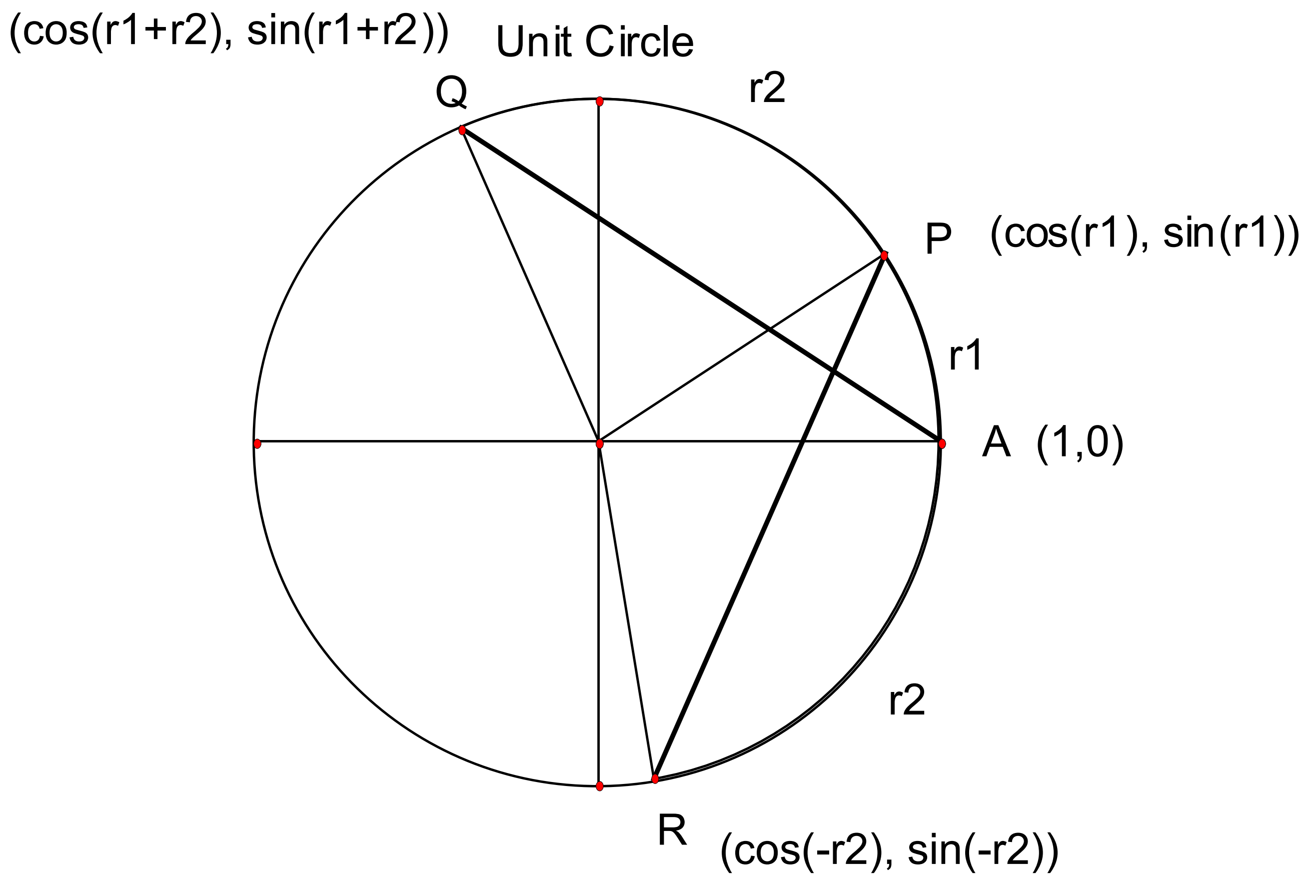
That's the geometric setup for our proof.

The angle measure of an included
The coordinates of A are (1, 0)

The coordinates of Q are (cos(r1+r2), sin(r1+r2))

The coordinates of P are (cos(r1), sin(r1))

The coordinates of R are (cos(-r2), sin(-r2))



Using the distance formula and setting $\overline{AQ} = \overline{PR}$ we have:

$$\sqrt{\left[\cos\left(r_1 + r_2\right) - 1\right]^2 + \left[\sin\left(r_1 + r_2\right) - 0\right]^2} =$$

$$\sqrt{\left[\cos\left(r_1\right) - \cos\left(-r_2\right)\right]^2 + \left[\sin\left(r_1\right) - \sin\left(-r_2\right)\right]^2}$$

Squaring both sides:

$$\left[\cos\left(r_1 + r_2\right) - 1\right]^2 + \sin^2\left(r_1 + r_2\right) =$$

$$\left[\cos\left(r_1\right) - \cos\left(r_2\right)\right]^2 + \left[\sin\left(r_1\right) + \sin\left(r_2\right)\right]^2$$

Expanding:

$$\cos^2(r_1+r_2)-2\cos(r_1+r_2)+1+\sin^2(r_1+r_2)=$$

$$\cos^2(r_1)-2\cos(r_1)\cos(r_2)+\cos^2(r_2)+$$

$$\sin^2(r_1)+2\sin(r_1)\sin(r_2)+\sin^2(r_2)$$

$$\left[\cos^2(r_1+r_2)\right]-2\cos(r_1+r_2)+1+\left[\sin^2(r_1+r_2)\right]=$$

$$\left[\cos^2(r_1)\right]-2\cos(r_1)\cos(r_2)+\left[\cos^2(r_2)\right]+$$

$$\left[\sin^2(r_1)\right]+2\sin(r_1)\sin(r_2)+\left[\sin^2(r_2)\right]$$

$$-2\cos(r_1+r_2)+2=$$

$$-2\cos(r_1)\cos(r_2)+1$$

$$+2\sin(r_1)\sin(r_2)+1$$

Subtracting 2 from each side and dividing by -2

$$\cos(r_1+r_2)=\cos(r_1)\cos(r_2)-\sin(r_1)\sin(r_2)$$

This is the cosine summation formulae

Plugging in -y for y and simplifying using odd/even identities

$$\cos\left(x-y\right)=\cos\left(x\right)\cos\left(y\right)+\sin\left(x\right)\sin\left(y\right)$$

The derivation for sin(x+y) is similar but won't be covered:

$$\sin(x+y)=\sin(x)\cos(y)+\sin(y)\cos(x)$$

and similarly

$$\sin(x-y)=\sin(x)\cos(y)-\sin(y)\cos(x)$$

To find tan(x+y):

$$\tan\left(x+y\right)=\frac{\sin\left(x+y\right)}{\cos\left(x+y\right)}=\frac{\sin x\cos y+\sin y\cos x}{\cos x\cos y-\sin x\sin y}$$

$$\frac{\sin x\cos y+\sin y\cos x}{\cos x\cos y-\sin x\sin y}\cdot\frac{\frac{1}{\cos x\cos y}}{\frac{1}{\cos x\cos y}}=\frac{\frac{\sin x}{\cos x}+\frac{\sin y}{\cos y}}{1-\frac{\sin x\sin y}{\cos x\cos y}}=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

so:

$$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$$

Similarly:

$$\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

Do additional problems on the handout.

Double Angle Formulas:

If

$$\sin(x+y)=\sin(x)\cos(y)+\sin(y)\cos(x)$$

then

$$\sin(x+x)=\sin(x)\cos(x)+\sin(x)\cos(x)$$

or simplified

$$\sin(2x) = 2\sin(x)\cos(x)$$

Similarly

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

Using the Pythagorean identity we get two other useful forms:

$$\cos(2x) = 2\cos^2(x) - 1$$

and

$$\cos(2x) = 1 - 2\sin^2(x)$$

The tangent double angle then becomes

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$$

Half Angle Formulas

Starting with

$$\cos(2x) = 2\cos^2(x) - 1$$

we substitute getting

$$\cos(x) = 2\cos^2\left(\frac{x}{2}\right) - 1$$

Solving for $\cos\left(\frac{x}{2}\right)$

We get

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}}$$

Starting with

$$\cos(2x) = 1 - 2\sin^2(x)$$

we substitute getting

$$\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right)$$

$$\sin\left(\frac{x}{2}\right)$$

We get

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$

Finally, there are three versions of the tangent half angle

$$\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{1+\cos(x)}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{\sin(x)}$$

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1+\cos(x)}$$

Example of half angle formula:

$$\sin\left(15^\circ\right) = \sin\left(\frac{30^\circ}{2}\right) = \pm\sqrt{\frac{1-\cos\left(30^\circ\right)}{2}} =$$

$$\pm\sqrt{\frac{1-\sqrt{3}/2}{2}} = \pm.258819$$

So which is it? .2588 or -.2588

Well 15° is in the first quadrant so .2588!