

## Derivative Formulas

### Simplifying how we find Derivatives

In our last class we found the derivatives of a few basic functions

For  $f(x) = c$  we have  $f'(x) = 0$

For  $f(x) = mx + b$  we have  $f'(x) = m$

For  $f(x) = x^2$  we have  $f'(x) = 2x$

For  $f(x) = \sqrt{x}$  we have  $f'(x) = \frac{-1}{2\sqrt{x}}$

And For  $f(x) = \sqrt{1-x^2}$  we have  $f'(x) = \frac{-x}{\sqrt{1-x^2}}$

It would be very tedious to have to use the derivative formula

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

every time we need to find a derivative.

So here we derive some general formulas or rules which make such calculations quite simple.

The derivative of a sum is the sum of a derivative

$$1. (f + g)'(x) = f'(x) + g'(x)$$

#### Proof

$$(f + g)'(x) = \lim_{h \rightarrow 0} \frac{(f + g)(x+h) - (f + g)(x)}{h} =$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - [f(x) + g(x)]}{h} =$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} =$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} =$$

$$f'(x) + g'(x)$$

The derivative of a constant times a function is the constant times the derivative

$$2. (Kf)'(x) = Kf'(x)$$

**Proof**

$$(Kf)'(x) = \lim_{h \rightarrow 0} \frac{Kf(x+h) - Kf(x)}{h} = K \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = Kf'(x)$$

**The Product Rule**

$$3. (f \cdot g)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

**Proof**

$$(f \cdot g)'(x) = \lim_{h \rightarrow 0} \frac{(f \cdot g)(x+h) - (f \cdot g)(x)}{h} =$$

$$\lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} =$$

Now we pull a rabbit out of our hat.

We add and subtract  $f(x+h)g(x)$  from the numerator.

$$\lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x) + f(x+h)g(x) - f(x+h)g(x)}{h} =$$

We rearrange and take out  $f(x+h)$  as a factor and also take out  $g(x)$  as a factor

$$\lim_{h \rightarrow 0} f(x+h) \frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} g(x) \frac{f(x+h) - f(x)}{h} =$$

Since  $f(x)$  is continuous at  $x$ , this becomes

$$f(x) \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} =$$

$$f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Using this theorem, it is easy to see **Power Rule** that if  $n$  is an integer  $> 0$

$$4. (x^n)' = n(x^{n-1})$$

We do this with a proof by induction.

For such a proof we have a series of statements  $P(n)$  where  $n=1,2,3\dots$

We need to show two things for a proof by induction

1. The anchor is true  $P(1)$
2. if we assume  $P(n)$  then  $P(n+1)$  is true.

### Proof

$P(1)$  is the statement  $(x^1)' = (x^0) = 1$

This is a simplification of

For  $f(x) = mx + b$  we have  $f'(x) = m$

Assuming that  $P(n)$  or  $(x^n)' = n(x^{n-1})$  is true

we want to show that  $P(n+1)$  or  $(x^{n+1})' = (n + 1)x^n$  is true.

We do this using the product rule

$$(x^{n+1})' = (x \cdot x^n)' = x(x^n)' + (x)'x^n = x \cdot nx^{n-1} + x^n = nx^n + x^n = (n + 1)x^n$$

### Example

If  $P(x) = 12x^3 - 6x - 2$  what is  $P(x)'$ ?

$$P(x)' = 12 \cdot 3x^2 - 6 = 36x^2 - 6$$

If  $P(x) = \frac{1}{4}x^4 - 2x^2 + x + 5$  what is  $P(x)'$ ?

$$P(x)' = \frac{1}{4} \cdot 4 \cdot x^3 - 2 \cdot 2x + 1 = x^3 - 4x + 1$$