Math 109 Calc 1 Lecture 7

Reciprocal and Quotient Rules

Review

Sum Rule

1. $(f+g)'(x) = f'(x) + g'(x)$

The Product Rule

2. $(f \cdot g)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

The Power Rule

3. $(x^n)' = n(x^{n-1})$

Reciprocal Rule

Assuming that g is differentiable we have

4. $\left(\frac{1}{g}\right)' = -\frac{g'(x)}{[g(x)]^2}$ for $g(x) \neq 0$

To show this we start with our usual definition of a derivative

$$\lim_{h\to 0} \frac{1}{h}\left[\frac{1}{g(x+h)} - \frac{1}{g(x)}\right] = \lim_{h\to 0} \frac{g(x)-g(x+h)}{h} \cdot \frac{1}{g(x)} \cdot \frac{1}{g(x+h)} =$$

$$-\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \cdot \frac{1}{g(x)} \cdot \frac{1}{g(x+h)} =$$

$$-g'(x) \lim_{h\to 0} \frac{1}{g(x)} \frac{1}{\underset{h\to 0}{g(x+h)}} = -\frac{g'(x)}{[g(x)]^2}$$

Example:

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{\mathbf{1}}{\sqrt{\boldsymbol{x}}} = \frac{\mathbf{1}}{\boldsymbol{g}(\boldsymbol{x})}$$

$$f'(x) = -\frac{g'(x)}{[g(x)]^2} = -\frac{\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} = -\frac{\frac{1}{2\sqrt{x}}}{x} = -\frac{1}{2\sqrt{x^3}}$$

Example:

Let n > 0 and let $p(x) = x^{-n} = \frac{1}{x^n}$ so, we have $g(x) = x^n$

Using the power and reciprocal formulas

$$p'(x) = \left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{[g(x)]^2} - \frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

Note: this shows that the power rule works for *n* an integer < 0

The Quotient Rule

5. $\left(\frac{f}{g}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ for $g(x) \neq 0$

I find this the hardest to remember. The key I think is the recall that the first term in the numerator is f'.
To show this rule we use the product and reciprocal rules

$$\left(\frac{f}{g}\right)' = \left[f(x)\frac{1}{\mathrm{g(x)}}\right]' = f(x)\left(\frac{1}{\mathrm{g(x)}}\right)' + \frac{1}{\mathrm{g(x)}}f'(x) =$$

$$\boldsymbol{f(x)\left(-\frac{g'(x)}{[g(x)]^2}\right) + \frac{f'(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

Example:

$$f(x) = \frac{5\mathrm{x}}{1 + \mathrm{x}}$$

$$\boldsymbol{f'(x)} = \frac{(5\mathrm{x})'(1 + \mathrm{x})\text{-}(5\mathrm{x})(1 + \mathrm{x})'}{(1 + \mathrm{x})^{\mathbf{2}}} = \frac{5(1 + \mathrm{x})\text{-}5\mathrm{x}}{(1 + \mathrm{x})^{\mathbf{2}}} = \frac{5}{(1 + \mathrm{x})^{\mathbf{2}}}$$

Example

$$f(x) = \frac{6x^2 - 1}{x^4 + 5x + 1}$$

What is $f'(x)$?

Here we must use the quotient rule

$$f'(x) = \frac{(6x^2 - 1)'(x^4 + 5x + 1) - (6x^2 - 1)(x^4 + 5x + 1)'}{(x^4 + 5x + 1)^2} =$$

$$\frac{12x(x^4 + 5x + 1) - (6x^2 - 1)(4x^3 + 5)}{(x^4 + 5x + 1)^2} =$$

$$\frac{12x^5 + 60x^2 + 12x - 24x^5 + 4x^3 - 30x^2 + 5}{(x^4 + 5x + 1)^2} =$$

$$\frac{-12x^5 + 4x^3 + 30x^2 + 12x \ + 5}{(x^4 + 5x + 1)^2}$$

$$f'(0) = \frac{5}{1} = 5$$